

CONVOLUTION EQUATIONS ON LATTICES: PERIODIC SOLUTIONS WITH VALUES IN A PRIME CHARACTERISTIC FIELD

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ABSTRACT. These notes are inspired by the theory of cellular automata. The latter aims, in particular, to provide a model for inter-cellular or inter-molecular interactions. A linear cellular automaton on a lattice Λ is a discrete dynamical system generated by a convolution operator $\Delta_a : f \mapsto f * a$ with kernel a concentrated in the nearest neighborhood ω of 0 in Λ . In [Za₁] we gave a survey (limited essentially to the characteristic 2 case) on the σ^+ -cellular automaton with kernel the constant function 1 in ω . In the present paper we deal with general convolution operators over a field of characteristic $p > 0$. Our approach is based on the harmonic analysis. We address the problem of determining the spectrum of a convolution operator in the spaces of pluri-periodic functions on Λ . This is equivalent to the problem of counting points on the associate algebraic hypersurface in an algebraic torus according to their torsion multi-orders. These problems lead to a version of the Chebyshev-Dickson polynomials parameterized this time by the set of all finite index sublattices of Λ and not by the naturals as in the classical case. It happens that the divisibility property of the classical Chebyshev-Dickson polynomials holds in this more general setting.

MP: - *Do you yourself perceive a fundamental difference between pure and applied mathematics?*

Stanislaw Ulam: - *I really don't. I think it's a question of language, and perhaps habits.*

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INTRODUCTION

These notes are inspired by the theory of linear cellular automata. Such an automaton on the integer lattice \mathbb{Z}^s can be viewed as a discrete dynamical system generated by a convolution operator $f \mapsto \Delta_a f = f * a$, acting on functions $f : \mathbb{Z}^s \rightarrow K$ with values in a Galois field $K = \text{GF}(p)$. Usually the kernel a of Δ_a is concentrated in the nearest neighborhood of $0 \in \mathbb{Z}^s$. We are interested more generally in systems of convolution equations

$$(1) \quad \Delta_{a_j} f = 0, \quad j = 1, \dots, t$$

with kernel $\bar{a} = (a_1, \dots, a_t)$ of bounded (and so finite) support. We address the following questions.

Problem. *Describe the set of all possible pluri-periods of the pluri-periodic solutions of (1). More precisely, given a pluri-period $\bar{n} \in \mathbb{N}^s$ compute the spectral multiplicities of (1) on the space of all \bar{n} -periodic functions on \mathbb{Z}^s and the dimension of the corresponding kernel $\ker \Delta_{\bar{a}}$.*

At present these problems seem to be out of reach. We provide however different interpretations that could be useful in future approaches.

Counting pluri-periods amounts to counting points on the associated affine algebraic variety $\Sigma_{\bar{a}}$ (called symbolic variety) according to their multi-orders. The symbolic variety $\Sigma_{\bar{a}}$ is a subvariety in the algebraic torus $(\bar{K}^\times)^s$, where \bar{K} stands for the algebraic closure of K . The multiplicative group \bar{K}^\times being a torsion group, the torus is covered by the finite subgroups. We are interested in the distribution of points on $\Sigma_{\bar{a}}$ according to the filtration of $(\bar{K}^\times)^s$ by finite subgroups.

The spectral multiplicities which appear in the above problem can be described via Chebyshev-Dickson polynomial systems. Such a system associates a degree d polynomial in $K[\lambda]$ to any sublattice of \mathbb{Z}^s of index d , namely the characteristic polynomial of (1) in the corresponding function space. The classical Chebyshev-Dickson polynomials appear in the simplest case where $s = 1$. In Theorem 0.1 we establish the divisibility property for Chebyshev-Dickson systems. Besides, we give in Proposition 2.11 a description of these systems via iterated resultants.

Using the harmonic analysis we interpret the points of $\Sigma_{\bar{a}}$ as \bar{a} -harmonic lattice characters; see Theorem 0.2 below. Here ' \bar{a} -harmonic' simply means 'satisfying (1)'. In the classical case, for a solution of (1) the value in a lattice point is a sum of its values over the neighbor points¹; this explains our terminology.

Resuming, in these notes we explore interplay between periods of solutions of a system of convolution equations on a lattice, on one hand, torsion orders of points on the associate symbolic variety, on the other hand, and harmonic characters. Let us develop along these lines in more detail.

¹In positive characteristic, one has to replace averaging by summation.

1. σ^+ -automaton. In [Za₁] we gave a survey on the σ^+ -automata on rectangular and toric grids. Let us recall the setup. On the integral lattice $\Lambda = \mathbb{Z}^s$ we consider the following function a^+ with values in the binary Galois field $\text{GF}(2)$:

$$(2) \quad a^+ = \delta_0 + \sum_{i=1}^s (\delta_{e_i} + \delta_{-e_i}),$$

where e_1, \dots, e_s stands for the canonical lattice basis. We let Δ_{a^+} denote the convolution operator $f \mapsto f * a^+$ acting on binary functions $f : \Lambda \rightarrow \text{GF}(2)$. It generates a discrete dynamical system on Λ called a σ^+ -automaton studied e.g., in [MOW], [Su], [GKW], [BR], [SB], [HMP]; see further references in [Za₁].

2. σ^+ -game. The σ^+ -automaton on the plane lattice $\Lambda = \mathbb{Z}^2$ is related to the solitaire game 'Lights Out', also called a σ^+ -game. Let us describe the game. Suppose that the offices in a department, which will be our table of game, correspond to the vertices of a grid $P_{m,n} = L_m \times L_n$, where L_m stands for the linear graph with m vertices. Suppose also that the interrupters are synchronized in such an uncommon way that turning off or on in one room changes automatically to the opposite the states in all rooms neighbors through a wall. The question arises whether the last person leaving the department can always manage to turn all the lights off.

It is possible to reduce this problem to an analogous one for the toric grid $\mathbb{T}_{m',n'} = C_{m'} \times C_{n'}$, where $m' = m + 1$, $n' = n + 1$ and C_m stands for the circular graph with m vertices. We let $\mathcal{F} = \mathcal{F}(\mathbb{T}_{m,n}, \text{GF}(2))$ be the function space on the torus equipped with the standard bilinear form $\langle \cdot, \cdot \rangle$. The move at a vertex v of $\mathbb{T}_{m,n}$ in the σ^+ -game, applied to a function (a 'pattern') $f \in \mathcal{F}(\mathbb{T}_{m,n}, \text{GF}(2))$, consists in the addition

$$f \mapsto f + a_v^+ \pmod{2},$$

where $a_v^+(u) = a^+(u + v)$ is the shifted star function (2) centered at $v \in \mathbb{T}_{m,n}$. Thus the σ^+ -game on the torus $\mathbb{T}_{m,n}$ is winning starting with the initial pattern f_0 if and only if f_0 can be decomposed into a sum of shifts of the star function a^+ .

The linear invariants of the σ^+ -game form a subspace $\mathcal{H} \subseteq \mathcal{F}$ orthogonal to all shifts a_v^+ , $v \in \mathbb{T}_{m,n}$. Indeed

$$h \in \mathcal{H} \iff \langle h, f + a_v^+ \rangle \equiv \langle h, f \rangle \pmod{2} \quad \forall v \in \mathbb{T}_{m,n}, \quad \forall f \in \mathcal{F}.$$

Moreover the initial pattern f_0 is winning if and only if $f_0 \in \mathcal{H}^\perp$. The functions $h \in \mathcal{H}$ are called harmonic [Za₁]. This is justified by the following property: for any vertex v of the grid $\mathbb{T}_{m,n}$, the value $h(v)$ is the sum modulo 2 of the values of h over the neighbors of v in $\mathbb{T}_{m,n}$. Actually $\mathcal{H} = \ker(\Delta_{a^+})$. Thus the σ^+ -game on a toric grid $\mathbb{T}_{m,n}$ is winning for any initial pattern if and only if $0 \notin \text{spec}(\Delta_{a^+})$. The latter is known to be equivalent to the condition $\gcd(T_m, T_n^+) = 1$, where T_m stands for the classical m th Chebyshev-Dickson polynomial over $\text{GF}(2)$ and $T_n^+(x) = T_n(x + 1)$, see [Za₁, 2.35] and the references therein.

3. This discussion leads to the following problems.

- Determine the set of all winning toric grids $\mathbb{T}_{m,n}$. Equivalently, determine the set of all pairs (m, n) such that the polynomials T_m and T_n^+ are coprime. Or, which is complementary, determine the set of all toric grids $\mathbb{T}_{m,n}$ admitting a nonzero binary harmonic function.

- Given $(m, n) \in \mathbb{N}^2$ compute the dimension $d(m, n)$ of the subspace \mathcal{H} of all harmonic functions on $\mathbb{T}_{m,n}$ or, equivalently, the dimension of the subspace \mathcal{H}^\perp of all winning patterns.

For m, n odd we provide several different interpretations of $d(m, n)$. In particular we will show that, over the algebraic closure \bar{K} of the base field $K = \text{GF}(2)$, there is an orthonormal basis of $\mathcal{H} \otimes \bar{K}$ consisting of harmonic characters on $\mathbb{T}_{m,n}$ with values in the multiplicative group \bar{K}^\times . An initial pattern f_0 on $\mathbb{T}_{m,n}$ is winning if and only if f_0 is orthogonal to all harmonic characters on $\mathbb{T}_{m,n}$. The latter ones are in one to one correspondence with the (m, n) -bi-torsion points on the symbolic hypersurface. In our case the symbolic hypersurface is the elliptic cubic in the torus $(\bar{K}^\times)^2 = (\mathbb{A}_{\bar{K}}^1 \setminus \{0\})^2$ with equation

$$(3) \quad x + x^{-1} + y + y^{-1} + 1 = 0.$$

Thus to determine all toric grids $\mathbb{T}_{m,n}$ admitting a nonzero binary harmonic function is the same as to determine all bi-torsion orders of points on the cubic (3), see [Za₁].

4. Linear cellular automata on abelian groups. In the present paper we consider similar problems for general linear cellular automata on abelian groups. Recall that the theory of cellular automata aims, in particular, to provide a model for inter-cellular or inter-molecular interactions. One can regard a linear cellular automaton on a group, or rather on the Caley graph of a group, as a discrete dynamical system generated by a convolution operator with kernel concentrated in a nearest neighborhood of the neutral element [MOW].

In more detail, suppose we are given a collection (a colony) of 'cells' placed at the vertices of a locally finite graph Γ . This determines the relation 'neighbors' for cells; neighbors can interact. Each cell can be in one of n cyclically ordered states; thus the state of the whole collection at a moment t is codified by a function $f_t : \Gamma \rightarrow \mathbb{Z}/n\mathbb{Z}$. In the subsequent portions of time, the cells simultaneously change their states. According to a certain local rule, the new state of a cell depends on the previous states of the given cell and of all its neighbors.

To define a cellular automaton, say, σ on Γ means to fix at each vertex v of Γ such a local rule, which does not depend on t . Such a collection of local rules determines a discrete dynamical system $\sigma : f_t \mapsto f_{t+1}$. Usually the edges $[v_0, v_1], \dots, [v_0, v_s]$ at v_0 are ordered. So the local rule at v_0 is a function $\phi_{v_0} : (\mathbb{Z}/n\mathbb{Z})^{s+1} \rightarrow \mathbb{Z}/n\mathbb{Z}$, and

$$(4) \quad f_{t+1}(v_0) = \phi_{v_0}(f_t(v_0), f_t(v_1), \dots, f_t(v_s)).$$

In the case where the graph Γ is homogeneous under a group action on Γ , it is natural to assume that the family of local rules is as well homogeneous. Consider for instance the Caley graph Γ of a finitely generated group G with a generating set $\{g_1, \dots, g_s\}$. Given a local rule ϕ_e for the neutral element $e \in G$, we can define ϕ_g for any vertex $g \in G$ as the shift of ϕ_e by g .

For an additive cellular automaton the local rule ϕ_e is a linear function. Consequently such an automaton is generated by a convolution operator $\Delta_a : f \mapsto f * a$ on G with kernel a supported in the nearest neighborhood of e . This kernel is just the coefficient function of ϕ_e . The evolution equation (4) can be written in this case as a heat equation

$$\partial_t(f_t) := f_{t+1} - f_t = \Delta_{a'}(f_t), \quad \text{where} \quad a' = a - \delta_e.$$

Here we restrict to additive cellular automata on lattices or toric grids, viewed as the Cayley graphs of finitely generated free abelian groups and finite abelian groups, respectively. In contrast with the classical setting, we allow distant interactions. So we deal with general convolution operators.

5. Convolution operators on lattices and Chebyshev-Dickson systems. For a field K of characteristic $p > 0$ and for a group G we let $\mathcal{F}(G, K)$ and $\mathcal{F}^0(G, K)$ denote the vector space of all functions $f : G \rightarrow K$, of all those with finite support, respectively. We consider the convolution

$$* : \mathcal{F}(G, K) \times \mathcal{F}^0(G, K) \ni (f, a) \longmapsto f * a \in \mathcal{F}(G, K),$$

where

$$f * a(g) = \sum_{h \in G} f(h)a(h^{-1}g) \quad \forall g \in G.$$

Fixing a we get the convolution operator $\Delta_a : f \longmapsto f * a$ acting on the space $\mathcal{F}(G, K)$. All such operators form a K -algebra $\text{Conv}_K(G)$ with $\Delta_{a_1} \circ \Delta_{a_2} = \Delta_{a_2 * a_1}$.

6. For a subgroup $H \subseteq G$ we let $\Delta_a|_H$ denote the restriction of Δ_a to the subspace $\mathcal{F}_H(G, K) \subseteq \mathcal{F}(G, K)$ of all H -periodic functions. Clearly $\mathcal{F}_H(G, K)$ is of finite dimension whenever H is of finite index in G , and $\dim \mathcal{F}_H(G, K) = [G : H]$.

In the sequel $G = \Lambda$ will be a lattice i.e., a free abelian group of finite rank. We let \mathcal{L} denote the set of all finite index sublattices in Λ . Ordered by inclusion, \mathcal{L} can be regarded as an ordered graph. Given a function $a \in \mathcal{F}^0(\Lambda, K)$ with finite support, we consider the spectra and the spectral multiplicities of $\Delta_a|_{\Lambda'}$, $\Lambda' \in \mathcal{L}$, in the algebraic closure \bar{K} of K . In particular we consider the function

$$(5) \quad d(a, \Lambda') = \dim \ker (\Delta_a|_{\Lambda'}), \quad \Lambda' \in \mathcal{L}.$$

The family of characteristic polynomials

$$\text{CharPoly}_{a, \Lambda'} = \text{CharPoly}(\Delta_a|_{\Lambda'}), \quad \Lambda' \in \mathcal{L},$$

will be called a Chebyshev-Dickson system. Recall that the n th Dickson polynomial $D_n(x, \alpha)$ over a finite field F is the unique polynomial verifying the identity $D_n(x + \alpha/x, \alpha) = x^n + \alpha^n/x^n$, where $\alpha \in F$. Whereas for $F = \text{GF}(2)$, the n th Chebyshev-Dickson polynomial of the first kind is $T_n(x) = D_n(x, 1)$. We recover the latter one as $T_n = \text{CharPoly}_{a, \Lambda'}$ when taking $K = \text{GF}(2)$, $a = a^+ - \delta_e$, $\Lambda = \mathbb{Z}$ and $\Lambda' = n\mathbb{Z}$.

The classical Chebyshev-Dickson system (T_n) possesses a number of interesting properties. It forms a commutative semigroup under composition that is, $T_n \circ T_m = T_{mn}$. Furthermore T_m divides T_n if $m \mid n$, moreover, $\gcd(T_m, T_n) = T_{\gcd(m, n)}$, etc. The composition property is not stable under shifts in the argument, and so does not hold in our more general setting. However, the divisibility property does hold. Namely we have the following

Theorem 0.1. *CharPoly $_{a, \Lambda''}$ divides CharPoly $_{a, \Lambda'}$ whenever $\Lambda' \subset \Lambda''$. Moreover, if the index of Λ' in Λ'' equals p^α then $\text{CharPoly}_{a, \Lambda'} = (\text{CharPoly}_{a, \Lambda''})^{p^\alpha}$.*

The second assertion allows to restrict to the subgraph $\mathcal{L}^0 \subseteq \mathcal{L}$ of all sublattices $\Lambda' \subseteq \Lambda$ with indices coprime with p . For a sublattice $\Lambda' \in \mathcal{L}^0$, the dimension $d(a, \Lambda')$ of the kernel of Δ_a equals the multiplicity of the zero root of the polynomial $\text{CharPoly}_{a, \Lambda'}$.

7. Systems of convolution equations. We let $K = \text{GF}(p)$ be the Galois field of characteristic $p > 0$, \bar{K} be the algebraic closure of K and \bar{K}^\times be the multiplicative group of \bar{K} . We fix a cortege $\bar{a} = (a_1, \dots, a_t)$ consisting of functions $a_j : \Lambda \rightarrow \bar{K}$ with bounded supports.

Given a lattice Λ we consider the system of convolution equations

$$(6) \quad \Delta_{a_j}(f) := f * a_j = 0, \quad j = 1, \dots, t, \quad \text{where } f : \Lambda \rightarrow \bar{K}.$$

We let $d(\bar{a}, \bar{n})$ denote the number of linearly independent \bar{n} -periodic solutions of (6). We call these solutions \bar{a} -harmonic.

We let

$$\text{CharPoly}_{\bar{a}, \Lambda'} = \gcd \left(\text{CharPoly}_{a_j, \Lambda'} : j = 1, \dots, t \right),$$

$$\ker(\Delta_{\bar{a}}|_{\Lambda'}) = \bigcap_{j=1}^t \ker(\Delta_{a_j}|_{\Lambda'}) \quad \text{and} \quad d(\bar{a}, \Lambda') = \dim \ker(\Delta_{\bar{a}}|_{\Lambda'}).$$

Thus $\text{CharPoly}_{\bar{a}, \Lambda'}(\lambda) = 0$ if and only if there exists a nonzero Λ' -periodic eigenfunction $f \in \mathcal{F}_{\Lambda'}(\Lambda, \bar{K})$ of $\Delta_{\bar{a}}$ with

$$\Delta_{a_j}(f) = \lambda \cdot f \quad \forall j = 1, \dots, t.$$

The set of zeros of $\text{CharPoly}_{\bar{a}, \Lambda'}$ counted with multiplicities is called the spectrum of $\Delta_{\bar{a}}|_{\Lambda'}$, and is denoted by $\text{spec}(\Delta_{\bar{a}}|_{\Lambda'})$. The set of spectra forms a graph Ξ ordered by inclusion. Due to the divisibility property in Chebyshev-Dickson systems, the map $\text{spec} : \mathcal{L} \rightarrow \Xi$ of ordered graphs is monotonous.

8. Symbolic variety. Given a basis $\mathcal{V} = (v_1, \dots, v_s)$ of Λ we can identify Λ with \mathbb{Z}^s . To a function $a : \Lambda = \mathbb{Z}^s \rightarrow \bar{K}$ with bounded support we associate the Laurent polynomial

$$\sigma_a = \sum_{u=(u_1, \dots, u_s) \in \mathbb{Z}^s} a(u) x^{-u} \in \bar{K}[x_1, x_1^{-1}, \dots, x_s, x_s^{-1}], \quad j = 1, \dots, t.$$

A cortege $\bar{a} = (a_1, \dots, a_t)$ determines an affine algebraic variety

$$\Sigma_{\bar{a}} = \{\sigma_{a_j} = 0 : j = 1, \dots, t\}$$

in the torus $(\bar{K}^\times)^s$. We call $\Sigma_{\bar{a}}$ the symbolic variety associated with the system (6).

The logarithm of the Weil zeta function counts the points on $\Sigma_{\bar{a}}$ over the Galois fields $\text{GF}(q)$, where $q = p^r$, $r \geq 0$. Whereas our purpose is to count, for every multi-index $\bar{n} = (n_1, \dots, n_s) \in \mathbb{N}^s$ with all n_i coprime to p , the number

$$d_{\bar{a}}(\bar{n}) = \text{card}(\Sigma_{\bar{a}, \bar{n}})$$

of \bar{n} -torsion points on the symbolic variety $\Sigma_{\bar{a}}$, where

$$\Sigma_{\bar{a}, \bar{n}} = \{\xi = (\xi_1, \dots, \xi_s) \in \Sigma_{\bar{a}} : \xi_i^{n_i} = 1, \quad i = 1, \dots, s\}.$$

Due to Theorem 0.2(a) below this same quantity arises in the spectral problem:

$$d_{\bar{a}}(\bar{n}) = d(\bar{a}, \bar{n}).$$

9. Harmonic characters. We let $\text{Char}(\Lambda, \bar{K}^\times)$ denote the set of all \bar{K}^\times -valued characters of Λ , that is of all homomorphisms $\chi : \Lambda \rightarrow \bar{K}^\times$. A character χ is called \bar{a} -harmonic if the function $\chi : \Lambda \rightarrow \bar{K}$ is; $\text{Char}_{\bar{a}\text{-harm}}(\Lambda, \bar{K}^\times)$ stands for the set of all \bar{a} -harmonic characters of Λ .

We denote by $\mathbb{N}_{\text{co}(p)}$ the set of all naturals coprime with p . Given a multi-index $\bar{n} \in \mathbb{N}_{\text{co}(p)}^s$ we consider the finite subgroup of the torus $(\bar{K}^\times)^s$

$$\mu_{\bar{n}} := \{\xi = (\xi_1, \dots, \xi_s) \in (\bar{K}^\times)^s : \xi_i^{n_i} = 1, i = 1, \dots, s\}.$$

Fixing a basis $\mathcal{V} = (v_1, \dots, v_s)$ of Λ we consider the product sublattices

$$\Lambda' = \Lambda_{\bar{n}, \mathcal{V}} = \sum_{i=1}^s n_i \mathbb{Z} v_i \subseteq \Lambda.$$

For such a sublattice $\Lambda' \subseteq \Lambda$, the Fourier transform provides a natural one to one correspondence between the set of all \bar{a} -harmonic characters of the quotient group $G = \Lambda/\Lambda'$ and the set of points on the symbolic variety with multi-torsion order dividing \bar{n} . Namely the following hold.

- Theorem 0.2.** (a) *For any sublattice $L' \in \mathcal{L}^0$, the subspace $\ker(\Delta_{\bar{a}}|_{\Lambda'})$ possesses an orthonormal basis of \bar{a} -harmonic characters. In particular there are $d(\bar{a}, \Lambda') = \text{mult}_{\lambda=0}(\text{CharPoly}_{\bar{a}, \Lambda'})$ such characters.*
- (b) *Fixing a basis \mathcal{V} of Λ provides a natural bijection $\text{Char}(\Lambda, \bar{K}^\times) \xrightarrow{\cong} (\bar{K}^\times)^s$. This bijection restricts to*

$$\text{Char}_{\bar{a}\text{-harm}}(\Lambda, \bar{K}^\times) \xrightarrow{\cong} \Sigma_{\bar{a}} \subseteq (\bar{K}^\times)^s.$$

Moreover $\forall \bar{n} \in \mathbb{N}_{\text{co}(p)}^s$ it further restricts to

$$\text{Char}_{\bar{a}\text{-harm}}(\Lambda/\Lambda_{\bar{n}, \mathcal{V}}, \bar{K}^\times) \xrightarrow{\cong} \Sigma_{\bar{a}, \bar{n}} := \Sigma_{\bar{a}} \cap \mu_{\bar{n}}.$$

In Section 1 we prove the second part of Theorem 0.1, see Corollary 1.5. The first parts of Theorems 0.1 and 0.2 are proven in Section 2, see Corollary 2.2 and Theorem 2.4, respectively. We also deduce an expression of polynomials in a Chebyshev-Dickson system via iterated resultants, see Proposition 2.11.

In Section 3 we prove Theorem 0.2(b) (see Proposition 3.1). Besides, we provide in Theorem 3.2 a dynamical criterion for existence of a nonzero periodic \bar{a} -harmonic function with a given pluri-period. Or what is the same, for existence of a point on the symbolic variety² with a given multi-torsion order. In addition we provide a formula for the orthogonal projection onto the space of all \bar{a} -harmonic functions.

In the final Section 4 we study similar problems over finite fields. Assuming p -freeness, in Theorem 4.4 we show that any \bar{a} -harmonic function with values in the original Galois field $\text{GF}(p)$ is a linear combination of traces of harmonic characters.

In Section 5 we discuss translation invariant subspaces generated by characters. The reader will also find here many concrete examples. In the final Section 6 he will find a discussion concerning the partnership graph, related to Zagier's theorem on finiteness of the connected components of this graph in the case of the σ^+ -automaton over the field $\text{GF}(2)$ on a plane lattice, see [Za1]. We indicate here several open problems.

The author is grateful to Don Zagier for clarifying discussions, in particular for the idea of processing in the present generality. Maxim Kontsevich suggested that the function $d_{\bar{a}} : \mathbb{N}^s \rightarrow \mathbb{N}$ should be studied via the technique of statistical sums within the framework on the Ising model. Our thanks also to Vladimir Berkovich for a kind assistance and to Dmitri Piontkovski for performing computer simulations.

²Which can be an arbitrary affine algebraic variety in a torus.

1. SYLOW p -SUBGROUPS AND CHEBYSHEV-DICKSON SYSTEMS

The Dickson polynomials $D_n(x, a)$ ($E_n(x, a)$, respectively) of the first (second) kind over a finite field of characteristic $p > 0$ can be characterized by the identities

$$\mu_1^n + \mu_2^n = D_n(\mu_1 + \mu_2, \mu_1\mu_2) \quad \text{resp.} \quad \mu_1^{n+1} - \mu_2^{n+1}/(\mu_1 - \mu_2) = E_n(\mu_1 + \mu_2, \mu_1\mu_2).$$

They also satisfy the relations [BZ]

$$D_{p^\alpha m} = (D_m)^{p^\alpha} \quad \text{resp.} \quad E_{p^\alpha m} = (E_m)^{p^\alpha}.$$

In this section we show that analogous relations hold for any Chebyshev-Dickson polynomial system $\text{CharPoly}_a : \mathcal{L} \rightarrow K[x]$ (see Corollary 1.5). Here K denotes a field of characteristic $p > 0$, Λ a lattice, $a \in \mathcal{F}^0(\Lambda, K)$ a K -valued function on Λ with bounded support and \mathcal{L} the ordered graph of all finite index sublattices $\Lambda' \subseteq \Lambda$, see §6 in the Introduction. This allows to recover CharPoly_a by its restriction to the subset $\mathcal{L}^0 \subseteq \mathcal{L}$ of all sublattices with indices coprime to p .

For a finite group G and a subset $A \subseteq G$ we let $\delta_A = \sum_{u \in A} \delta_u$ denote the characteristic function of A , where δ_u stands for the delta-function on G concentrated on $u \in G$. For a function $a = \sum_{u \in G} a(u)\delta_u$ on G we let

$$|a|A| = \sum_{u \in A} a(u), \quad \text{CharPoly}_{a,G} = \det(\Delta_a - \lambda \cdot 1) \quad \text{and} \quad d(a, G) = \dim \ker(\Delta_a).$$

The following Pushforward Lemma is simple, so we leave the proof to the reader.

Lemma 1.1. *For a normal subgroup $H \subseteq G$ and for any $a \in \mathcal{F}^0(G, K)$, the function*

$$(7) \quad a_*(v + H) = |a|(v + H)| = \sum_{v' \in H} a(v + v')$$

is a unique function in $\mathcal{F}^0(G/H, K)$ satisfying $a_ \circ \pi = a * \delta_H$, where $\pi : G \twoheadrightarrow G/H$ is the canonical surjection. Moreover, there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{F}_H(G, K) & \xrightarrow{\Delta_a} & \mathcal{F}_H(G, K) \\ \downarrow \cong & & \downarrow \cong \\ \mathcal{F}(G/H, K) & \xrightarrow{\Delta_{a_*}} & \mathcal{F}(G/H, K), \end{array}$$

where

$$\mathcal{F}_H(G, K) = \{f \in \mathcal{F}(G, K) : \tau_h(f) = f \quad \forall h \in H\} \quad \text{and} \quad \tau_h(f)(g) = f(gh).$$

The convolution algebra $\text{Conv}_K(G)$ consists of all operators on the space $\mathcal{F}^0(L, K)$ commuting with shifts. Moreover the shifts $(\tau_u : u \in G)$ generate $\text{Conv}_K(G)$ as a K -vector space. Indeed $\forall a \in \mathcal{F}^0(G, K)$ one has

$$(8) \quad \Delta_a = \sum_{g \in G} a(g)\tau_{g^{-1}}.$$

Notice that $\tau_{g^{-1}} = \Delta_{\delta_g}$ and $a = \Delta_a(\delta_e)$, where $e \in G$ denotes the neutral element.

For an endomorphism $A \in \text{End}(\mathbb{A}_K^n)$ we let $A = S_A + N_A$ be the Jordan decomposition, where S_A is semisimple, N_A nilpotent and S_A, N_A commute. It is defined over the algebraic closure \bar{K} of K .

Proposition 1.2. *Let $G = F \times H$ be a direct product of two abelian groups, where $H = \bigoplus_{i=1}^n \mathbb{Z}/p^{r_i}\mathbb{Z}$. Then for any $a \in \mathcal{F}^0(G, K)$ the following hold.*

- (a) $S_{\Delta_a} = S_{\Delta_{a_*}} \otimes 1_H$, where $a_* \in \mathcal{F}(F, K)$ is as in (7) above.
- (b) $\text{CharPoly}_{a,G} = (\text{CharPoly}_{a_*,F})^{\text{ord } H}$.

Consequently there exists a nonzero a -harmonic function on G if and only if there is a nonzero a_* -harmonic function on F .

Proof. (a) follows by induction on n , once it is established for $n = 1$. Letting $H = \mathbb{Z}/p^r\mathbb{Z}$ we will show that

$$(9) \quad \Delta_a^{p^r} = \Delta_{a_*}^{p^r} \otimes 1_H.$$

To this end, decomposing $u \in G$ as $u = u' + u''$ with $u' \in F$ and $u'' \in H$, we obtain

$$\begin{aligned} \Delta_a^{p^r} &= \sum_{u \in G} [a(u)]^{p^r} (\tau_{-u})^{p^r} = \sum_{u' \in F} \left(\sum_{u'' \in H} [a(u' + u'')]^{p^r} \right) (\tau_{-u'})^{p^r} \\ &= \left(\sum_{u' \in F} a_*(u') \tau_{-u'} \right)^{p^r} \otimes 1_H = \Delta_{a_*}^{p^r} \otimes 1_H. \end{aligned}$$

This proves (9). By virtue of (9) we get

$$S_{\Delta_a}^{p^r} + N_{\Delta_a}^{p^r} = S_{\Delta_{a_*}}^{p^r} \otimes 1_H + N_{\Delta_{a_*}}^{p^r} \otimes 1_H.$$

By the uniqueness of the Jordan decomposition we have $S_{\Delta_a}^{p^r} = S_{\Delta_{a_*}}^{p^r} \otimes 1_H$ and so (a) follows. Now (b) is immediate from (a). \square

From Proposition 1.2 we deduce the following corollaries.

Corollary 1.3. *If $G = \bigoplus_{i=1}^k \mathbb{Z}/p^{r_i}\mathbb{Z}$, where $p = \text{char } K$, then*

$$\text{CharPoly}_{a,G} = (x - |a|)^{\text{ord } G}.$$

In particular there exists an a -harmonic function on G if and only if the constant function 1 on G is a -harmonic, if and only if $|a| = 0$.

Remark 1.4. Letting $K = \text{GF}(2)$, $G = \mathbb{Z}/4\mathbb{Z}$ and $a = a^+ = \delta_0 + \delta_1 + \delta_{-1} \in \mathcal{F}(G, K)$ we obtain $\text{CharPoly}_{a,G} = (x + 1)^4$. Hence the algebraic multiplicity of the spectral value $\lambda = 1$ of Δ_a equals 4. Although $S_{\Delta_a} = 1_G$, the nilpotent part N_{Δ_a} is present and $\dim \ker(\Delta_a + 1_G) = 2$.

Given $\bar{a} = (a_1, \dots, a_t) \in (\mathcal{F}^0(G, K))^t$ we let as before

$$\ker(\Delta_{\bar{a}}) = \bigcap_{j=1}^t \ker(\Delta_{a_j}), \quad d(\bar{a}, G) = \dim \ker(\Delta_{\bar{a}})$$

and

$$\text{CharPoly}_{\bar{a},G} = \text{CharPoly}(\Delta_{\bar{a}}) = \gcd(\text{CharPoly}(\Delta_{a_j}) : j = 1, \dots, t).$$

For a subgroup $H \subseteq G$ we let

$$\text{CharPoly}_{\bar{a}_*,G/H} = \text{CharPoly}(\Delta_{\bar{a}}|_{\mathcal{F}_H(G, K)}),$$

where $\bar{a}_* = ((a_1)_*, \dots, (a_n)_*) \in \mathcal{F}(G/H, K)$.

Corollary 1.5. *Let Λ be a lattice and $\Lambda' \subseteq \Lambda$ a sublattice of index $p^\alpha q$, where $q \not\equiv 0 \pmod p$. Then there exists a unique intermediate sublattice Λ'' of index q in Λ , where $\Lambda' \subseteq \Lambda'' \subseteq \Lambda$. Moreover $\forall \bar{a} \in (\mathcal{F}^0(\Lambda, K))^t$ one has*

$$(10) \quad \text{CharPoly}_{\bar{a}, \Lambda'} = (\text{CharPoly}_{\bar{a}, \Lambda''})^{p^\alpha}.$$

Proof. It is enough to show (10) for $t = 1$; then it follows easily for any $t \geq 1$. So letting $t = 1$ and $a_1 = a$ we decompose

$$G = \Lambda/\Lambda' = F \oplus G(p),$$

where $G(p)$ is the Sylow p -subgroup of G and $\text{ord } F = q$. We let $\Lambda'' = \pi'^{-1}(G(p))$, where $\pi' : \Lambda \twoheadrightarrow G$, so that $\Lambda/\Lambda'' \cong F$. Due to the Pushforward Lemma 1.1,

$$(11) \quad \text{CharPoly}_{a, \Lambda'} = \text{CharPoly}_{\pi'_* a, G} \quad \text{and} \quad \text{CharPoly}_{a, \Lambda''} = \text{CharPoly}_{\pi''_* a, F},$$

where $\pi'' : \Lambda \twoheadrightarrow F$. Now (10) follows from (11) in view of Proposition 1.2(b). \square

2. CHEBYSHEV-DICKSON SYSTEMS IN THE p -FREE CASE

2.1. Naive Fourier transform on convolution algebras. For a finite group G there are natural isomorphisms

$$(\mathcal{F}(G, K), *) \xrightarrow{\varphi} \text{Conv}_K(G) \xrightarrow{\psi} K[G],$$

where $\varphi : a \mapsto \Delta_a$, $\text{Conv}_K(G)$ is the convolution algebra and $K[G]$ is the group algebra of G over K . For instance [MOW], the group ring of a finite abelian group

$$G = \bigoplus_{i=1}^s \mathbb{Z}/n_i \mathbb{Z}$$

is the truncated polynomial ring

$$K[G] = \bigotimes_{i=1}^s K[x_i]/(x_i^{n_i} - 1).$$

The ideals of $\mathcal{F}(G, K)$ are called convolution ideals. In particular, for any subgroup $H \subseteq G$, the translation invariant subspace

$$\mathcal{F}_H(G, K) = \{f \in \mathcal{F}(G, K) : \tau_h(f) = f \quad \forall h \in H\}$$

is a convolution ideal.

The composition $\psi \circ \varphi : \mathcal{F}(G, K) \rightarrow K[G]$, $a \mapsto \tilde{a}$, provides a naive Fourier transform, which sends Δ_a to the operator of multiplication by \tilde{a} in $K[G]$ and $\ker(\Delta_a)$ to the annihilator ideal $\text{Ann}(\tilde{a})$, where $(\tilde{a}) \subseteq K[G]$ is the principal ideal generated by \tilde{a} . Thus G possesses a nonzero a -harmonic function if and only if $\tilde{a} \in K[G]$ is a zero divisor.

The next result follows immediately from the Burnside Theorem. Alternatively it can be deduced using the Fourier transform, see below.

Lemma 2.1. *For any finite abelian group G of order coprime to p the following hold.*

- (a) $\mathcal{F}(G, \bar{K})$ admits a decomposition into a direct sum of one-dimensional $\text{Conv}_{\bar{K}}(G)$ -submodules generated by characters:

$$\mathcal{F}(G, \bar{K}) = \bigoplus_{g^\vee \in G^\vee} (g^\vee).$$

- (b) Any convolution ideal $I \subseteq \mathcal{F}(G, \bar{K})$ is principal, generated by the sum of characters contained in I . Furthermore there is a decomposition

$$\mathcal{F}(G, \bar{K}) = I \oplus \text{Ann}(I).$$

In particular, for any subgroup $H \subseteq G$ one has

$$\mathcal{F}(G, \bar{K}) = \mathcal{F}_H(G, \bar{K}) \oplus \text{Ann}(\mathcal{F}_H(G, \bar{K})),$$

where

$$\mathcal{F}_H(G, \bar{K}) = \sum_{H \subseteq \ker(g^\vee)} (g^\vee).$$

2.2. Dual group and Fourier transform. In the sequel we let $K = \text{GF}(p^r)$. Thus the multiplicative group \bar{K}^\times is a torsion group, with torsion orders coprime with p . We let G be a finite abelian group of order coprime with p , and $\mathbb{N}_{\text{co}(p)}$ be the set of all positive integers coprime to p .

The field \bar{K} contains all roots of unity with orders dividing $\text{ord } G$. Hence the characters of G can be realized as homomorphisms $G \rightarrow \bar{K}^\times$. This defines a natural embedding $G^\vee \hookrightarrow \mathcal{F}(G, \bar{K})$.

The Fourier transform $F : \mathcal{F}(G, \bar{K}) \rightarrow \mathcal{F}(G^\vee, \bar{K})$ is defined as usual [Nic] via

$$(12) \quad F : f \longmapsto \hat{f}, \quad \text{where} \quad \hat{f}(g^\vee) = \sum_{g \in G} f(g) g^\vee(g), \quad g^\vee \in G^\vee.$$

Its inverse $F^{-1} : \mathcal{F}(G^\vee, \bar{K}) \rightarrow \mathcal{F}(G, \bar{K})$ can be defined via

$$F^{-1} : \varphi \longmapsto \hat{\varphi}, \quad \text{where} \quad \hat{\varphi}(g) = \frac{1}{\text{ord}(G)} \sum_{g^\vee \in G^\vee} \varphi(g^\vee) g^\vee(g^{-1}), \quad g \in G.$$

With this notation $\hat{\hat{f}} = f$ and $\hat{\hat{\varphi}} = \varphi$, which does not lead to a confusion as we never exploit the Fourier transform on the dual group G^\vee .

Up to constant factors, both F and F^{-1} send δ -functions to characters and vice versa. Namely $\forall g \in G, \forall g^\vee \in G^\vee$,

$$\widehat{\delta_g} = g, \quad \widehat{g} = \delta_g \quad \text{and} \quad \widehat{\delta_G} = \sum_{g \in G} \widehat{\delta_g} = \text{ord}(G) \delta_{e^\vee},$$

respectively,

$$\text{ord}(G) \widehat{\delta_{g^\vee}} = (g^\vee)^{-1}, \quad \widehat{g^\vee} = \text{ord}(G) \delta_{(g^\vee)^{-1}} \quad \text{and} \quad \widehat{\delta_{G^\vee}} = \delta_e.$$

Furthermore F sends the convolution in the ring $\mathcal{F}(G, \bar{K})$ into the pointwise multiplication on $\mathcal{F}(G^\vee, \bar{K})$ giving an isomorphism of \bar{K} -algebras. The convolution operator Δ_a is sent to the operator of multiplication by \hat{a} . The Fourier transform of a character being proportional to a delta-function, any character $g^\vee \in G^\vee$, up to a scalar factor, is a convolution idempotent, and any convolution idempotent of $(\mathcal{F}(G, \bar{K}), *)$ is proportional to a sum of characters.

In the \bar{K} -vector space $\mathcal{F}(G, \bar{K})$ we consider the non-degenerate symmetric bilinear form

$$\langle f_1, f_2 \rangle_1 = \frac{1}{\text{ord}(G)} \sum_{g \in G} f_1(g) f_2(g^{-1}) = \frac{1}{\text{ord}(G)} f_1 * f_2(e).$$

Its Fourier dual is the following bilinear form in $\mathcal{F}(G^\vee, \bar{K})$:

$$\langle \varphi_1, \varphi_2 \rangle_2 = \frac{1}{(\text{ord}(G))^2} \sum_{g^\vee \in G^\vee} \varphi_1(g^\vee) \varphi_2(g^\vee).$$

Indeed we have $\langle \widehat{f_1}, \widehat{f_2} \rangle_2 = \langle f_1, f_2 \rangle_1$. The characters $(g^\vee : g^\vee \in G^\vee)$ form an orthonormal basis in $\mathcal{F}(G, \bar{K})$ with respect to the form $\langle \cdot, \cdot \rangle_1$.

2.3. Divisibility in Chebyshev-Dickson systems. In this section G denotes a finite abelian group of order coprime to p . For a t -tuple $\bar{a} = (a_1, \dots, a_t) \in (\mathcal{F}^0(G, \bar{K}))^t$ we let as before $\ker(\Delta_{\bar{a}}) = \bigcap_{j=1}^t \ker(\Delta_{a_j})$ and

$$\text{CharPoly}_{\bar{a}, G} = \text{CharPoly}(\Delta_{\bar{a}}) = \gcd(\text{CharPoly}(\Delta_{a_j}) : j = 1, \dots, t).$$

By the Pushforward Lemma 1.1, for a subgroup $H \subseteq G$ we have

$$\text{CharPoly}_{\bar{a}, G/H} = \text{CharPoly}(\Delta_{\bar{a}} | \mathcal{F}_H(G, \bar{K})).$$

We let

$$V(a_j) = \hat{a}_j^{-1}(0) \subseteq G^\vee \quad \text{and} \quad V(\bar{a}) = \bigcap_{j=1}^t V(a_j) \subseteq G^\vee.$$

We also let $\text{Char}_{\bar{a}\text{-harm}}(G, \bar{K}^\times)$ denote the set of all \bar{a} -harmonic characters of G . The following corollary is straightforward from Lemma 2.1.

Corollary 2.2. (a) *For any $a \in \mathcal{F}^0(G, \bar{K})$, the characters form an orthonormal basis $\mathcal{F}(G, \bar{K})$ of eigenfunctions of Δ_a , the matrix of Δ_a in this basis is diagonal and so*

$$\text{CharPoly}(\Delta_a) = \prod_{g^\vee \in G^\vee} (x - \hat{a}(g^\vee)).$$

$$\text{Consequently} \quad \text{spec}(\Delta_a) = \hat{a}(G^\vee) \subseteq \bar{K}, \quad \ker(\Delta_a) = \left(\widehat{\delta_{V(a)}} \right) \text{ and}$$

$$d(a, G) = \text{card } V(a) = \text{mult}_0 \text{CharPoly}(\Delta_a).$$

(b) *Similarly for any $\bar{a} = (a_1, \dots, a_t) \in (\mathcal{F}^0(G, \bar{K}))^t$, the \bar{a} -harmonic characters form an orthonormal basis in $\ker(\Delta_{\bar{a}})$,*

$$\ker(\Delta_{\bar{a}}) = \left(\widehat{\delta_{V(\bar{a})}} \right) \quad \text{and} \quad d(\bar{a}, G) = \text{card } V(\bar{a}).$$

Moreover there is a bijection $V(\bar{a}) \cong \text{Char}_{\bar{a}\text{-harm}}(G, \bar{K}^\times)$. Consequently the group G admits a nonzero \bar{a} -harmonic function if and only if it admits an \bar{a} -harmonic character.

A convolution ideal $I \subseteq \mathcal{F}(G, \bar{K})$ and the annihilator ideal $\text{Ann}(I)$ being $\Delta_{\bar{a}}$ -stable (see Lemma 2.1), the Pushforward Lemma 1.1 yields the following result.

Proposition 2.3. *For any convolution ideal $I \subseteq \mathcal{F}(G, \bar{K})$, any subgroup $H \subseteq G$ and any $\bar{a} \in (\mathcal{F}^0(G, \bar{K}))^t$ one has*

$$(13) \quad \text{CharPoly}(\Delta_{\bar{a}} | I) \mid \text{CharPoly}(\Delta_{\bar{a}}) \quad \text{and} \quad \text{CharPoly}_{\bar{a}, G/H} \mid \text{CharPoly}_{\bar{a}, G}.$$

Now we can readily deduce the divisibility property in Chebyshev-Dickson systems disregarding the assumption of p -freeness.

Theorem 2.4. *For any $\Lambda_1, \Lambda_2 \in \mathcal{L}$ we have:*

$$\text{CharPoly}_{\bar{a}, \Lambda_1 + \Lambda_2} \mid \gcd(\text{CharPoly}_{\bar{a}, \Lambda_1}, \text{CharPoly}_{\bar{a}, \Lambda_2})$$

and

$$\text{lcm}(\text{CharPoly}_{\bar{a}, \Lambda_1}, \text{CharPoly}_{\bar{a}, \Lambda_2}) \mid \text{CharPoly}_{\bar{a}, \Lambda_1 \cap \Lambda_2}.$$

Consequently $\text{CharPoly}_{\bar{a}, \Lambda_2} \mid \text{CharPoly}_{\bar{a}, \Lambda_1}$ whenever $\Lambda_1 \subseteq \Lambda_2$.

Proof. It suffices to show the latter assertion, then the former ones follow immediately. For a chain of finite index sublattices $\Lambda' \subseteq \Lambda'' \subseteq \Lambda$ we let $G = \Lambda/\Lambda'$, $H = \Lambda''/\Lambda'$ so that $G/H = \Lambda/\Lambda''$, and

$$\bar{a}' = \pi'_* \bar{a} \in \mathcal{F}(G, \bar{K}), \quad \bar{a}'' = \pi''_* \bar{a} \in \mathcal{F}(G/H, \bar{K}),$$

where $\pi : G \twoheadrightarrow G/H$, $\pi' : \Lambda \twoheadrightarrow G$ and $\pi'' = \pi \circ \pi' : \Lambda \twoheadrightarrow G/H$.

We assume first that the index of Λ' in Λ is coprime with p and so $\Lambda', \Lambda'' \in \mathcal{L}^0$. By the Pushforward Lemma 1.1 we obtain

$$\text{CharPoly}_{\bar{a}, \Lambda'} = \text{CharPoly}_{\bar{a}', G} \quad \text{and} \quad \text{CharPoly}_{\bar{a}, \Lambda''} = \text{CharPoly}_{\bar{a}'', G/H}.$$

Hence by (13)

$$(14) \quad \text{CharPoly}_{\bar{a}, \Lambda''} \mid \text{CharPoly}_{\bar{a}, \Lambda'}.$$

In the general case, assuming that $\Lambda_1 \subseteq \Lambda_2$ we consider the decomposition $G_1 = \Lambda/\Lambda_1 = F \oplus G_1(p)$, where $G_1(p) \subseteq G_1$ is the Sylow p -subgroup. Letting $G_2 = \Lambda_2/\Lambda_1 \subseteq G_1$ we obtain $G_2(p) = G_1(p) \cap G_2$. For the sublattices $\Lambda_i'' = \pi^{-1}(G_i(p)) \subseteq \Lambda$, $i = 1, 2$, where $\pi : \Lambda \twoheadrightarrow G_1$, we have $\Lambda_i'' \supseteq \Lambda_i$ and $\Lambda_1'', \Lambda_2'' \in \mathcal{L}^0$. Since also $\Lambda_1'' \subseteq \Lambda_2''$, by virtue of (14) we obtain $\text{CharPoly}_{\bar{a}, \Lambda_2''} \mid \text{CharPoly}_{\bar{a}, \Lambda_1''}$. By (10) $\text{CharPoly}_{\bar{a}, \Lambda_i} = (\text{CharPoly}_{\bar{a}, \Lambda_i''})^{p^{\alpha_i}}$, $i = 1, 2$, where $\alpha_2 \leq \alpha_1$. Now the result follows. \square

Remark 2.5. Letting $\Lambda'' = \Lambda$ we deduce that $(x - |a|) \mid \text{CharPoly}_{\bar{a}, \Lambda'} \quad \forall \Lambda' \in \mathcal{L}, \quad \forall a \in \mathcal{F}^0(\Lambda, \bar{K})$. We note that the eigenspace in $\mathcal{F}(\Lambda, \bar{K})$, which corresponds to the eigenvalue $|a|$ of Δ_a , contains the constant function 1; cf. Corollary 1.3.

Examples 2.6. 1. Letting $G = G_1 \times G_2$ and $a = a_1 \otimes a_2 \in \mathcal{F}(G, \bar{K})$, where $a_i \in \mathcal{F}(G_i, \bar{K})$, $i = 1, 2$, we obtain

$$\text{CharPoly}(\Delta_a) = \prod_{i,j} (x - \lambda_i \mu_j),$$

where $\lambda_1, \dots, \lambda_{\text{ord}(G_1)}$ and $\mu_1, \dots, \mu_{\text{ord}(G_2)}$ denote the eigenvalues of Δ_{a_1} and Δ_{a_2} , respectively.

2. Similarly, letting $a = a_1 \otimes 1 \oplus 1 \otimes a_2 \in \mathcal{F}(G, \bar{K})$, where again $G = G_1 \times G_2$, we obtain

$$\text{CharPoly}(\Delta_a) = \prod_{i,j} (x - (\lambda_i + \mu_j)).$$

The latter formula applies in particular to the graph Laplacians with kernels $a_i^- = a_{G_i}^+ - \delta_0$, $i = 1, 2$ and $a^- = a_G^+ - \delta_0$, respectively, where a^+ is the star-function as in (2). In the characteristic 2 case cf. Bacher's Lemma 2.10(a) in [Za1].

2.4. Symbol of a convolution operator.

Definition 2.7. Fixing a lattice Λ of rank s , a basis $\mathcal{V} = (v_1, \dots, v_s)$ of Λ and an n -tuple $\bar{n} = (n_1, \dots, n_s)$, where $n_i \in \mathbb{N}$, we let

$$\Lambda_{\bar{n}, \mathcal{V}} = \sum_{i=1}^s n_i \mathbb{Z} v_i \cong \bigoplus_{i=1}^s n_i \mathbb{Z}$$

be the product sublattice of Λ generated by $n_1 v_1, \dots, n_s v_s$. There is an isomorphism of K -algebras

$$\sigma_{\mathcal{V}} : \text{Conv}_K(\Lambda) \xrightarrow{\cong} K[x_1, x_1^{-1}, \dots, x_s, x_s^{-1}], \quad \Delta_a \longmapsto \sigma_{a, \mathcal{V}},$$

which associates to a convolution operator Δ_a on Λ its \mathcal{V} -symbol, that is the Laurent polynomial in s variables

$$(15) \quad \sigma_{a, \mathcal{V}} = \sum_{v = \sum_{i=1}^s \alpha_i e_i \in \Lambda} a(v) x^{-\alpha(v)} = \sum_{v = \sum_{i=1}^s \alpha_i e_i \in \Lambda} a(v) x_1^{-\alpha_1} \cdots x_s^{-\alpha_s}.$$

The inverse $\sigma_{\mathcal{V}}^{-1}$ is given by

$$x_i^{-1} \longmapsto \tau_{v_i} \quad \text{and} \quad x_i \longmapsto \tau_{-v_i}, \quad i = 1, \dots, s.$$

The algebraic hypersurface in the s -torus

$$(16) \quad \Sigma_{a, \mathcal{V}} = \sigma_{a, \mathcal{V}}^{-1}(0) \subseteq (K^\times)^s$$

associate with Δ_a will be called the symbolic hypersurface. Similarly, for a sequence of convolution operators $\Delta_{\bar{a}} = (\Delta_{a_j} : j = 1, \dots, t)$ its symbolic variety is the affine subvariety in the torus

$$(17) \quad \Sigma_{\bar{a}, \mathcal{V}} = \bigcap_{j=1}^t \sigma_{a_j, \mathcal{V}}^{-1}(0) \subseteq (K^\times)^s.$$

Example 2.8. (see [Za₁]) For $K = \text{GF}(2)$, $\Lambda = \mathbb{Z}^2$, $\mathcal{V} = (e_1, e_2)$ and $a = a^+$ the symbolic hypersurface $\Sigma_{a^+, \mathcal{V}}$ is the elliptic cubic (3) in $(\bar{K}^\times)^2$. Alternatively, this cubic can be given by the equation

$$x^2 y + x y^2 + x y + x + y = 0.$$

For a finite abelian group $G = \mathbb{Z}_{\bar{n}} = \bigoplus \mathbb{Z}/n_i \mathbb{Z}$, where $\bar{n} = (n_1, \dots, n_s) \in \mathbb{N}_{\text{co}(p)}^s$, we let $\mathcal{U} = (e_1, \dots, e_s)$ denote the standard base of G . We let also

$$\mu_{\bar{n}} = \bigoplus_{i=1}^s \mu_{n_i} \subseteq (\bar{K}^\times)^s,$$

where $\mu_n \subseteq \bar{K}^\times$ stands for the cyclic group of n th roots of unity. The correspondence

$$g^\vee \longmapsto (g^\vee(e_1), \dots, g^\vee(e_s))$$

yields an isomorphism

$$\varphi : G^\vee \xrightarrow{\cong} \mu_{\bar{n}}.$$

This isomorphism relates the symbol of a convolution operator Δ_a and the Fourier transform \hat{a} of its kernel.

Lemma 2.9. *For any $a \in \mathcal{F}(G, \bar{K})$ we have*

$$\hat{a} = (\sigma_{a,\mathcal{V}} | \mu_{\bar{n}}) \circ \varphi.$$

Consequently

$$(18) \quad \text{CharPoly}_{a,\bar{n},\mathcal{V}} := \text{CharPoly}_{a,\Lambda_{\bar{n},\mathcal{V}}} = \prod_{\xi \in \mu_{\bar{n}}} (x - \sigma_{a,\mathcal{V}}(\xi)).$$

Proof. For any $g \in G, g^\vee \in G^\vee$, letting $\xi_i = g^\vee(e_i)$, $i = 1, \dots, s$, by (8) and (15) we obtain:

$$\begin{aligned} \hat{a}(g) \cdot g^\vee(g) &= \Delta_a(g^\vee)(g) = \left(\sum_{v \in G} a(v) \tau_{-v} \right) (g^\vee)(g) = \sum_{v \in G} a(v) \tau_{-v}(g^\vee)(g) \\ &= \sum_{v = \sum_{j=1}^s \alpha_j e_j \in G} a(v) g^\vee(g - v) = \sigma_{a,\mathcal{V}}(\xi_1, \dots, \xi_s) \cdot g^\vee(g) = \sigma_{a,\mathcal{V}}(\xi), \end{aligned}$$

where $\xi = (\xi_1, \dots, \xi_s) \in \mu_{\bar{n}}$. Indeed

$$g^\vee(g - v) = g^\vee(g) g^\vee(-v) = g^\vee(g) g^\vee \left(- \sum_{j=1}^s \alpha_j e_j \right) = g^\vee(g) \prod_{j=1}^s \xi_j^{-\alpha_j}.$$

The equality (18) follows now from Proposition 2.2. \square

2.5. Chebyshev-Dickson systems and iterated resultants.

Definition 2.10. We consider a Laurent polynomial $\Omega = P/y^\alpha$, where $P \in \bar{K}[y_1, \dots, y_s]$ is a polynomial coprime with $y^\alpha = y_1^{\alpha_1} \dots y_s^{\alpha_s}$. Given a multi-index $\bar{n} = (n_1, \dots, n_s) \in \mathbb{N}^s$, we define recursively the iterated resultant $\text{res}_{\bar{n}}(\Omega) := Q_s \in \bar{K}[x]$ via

$$Q_0(x, y_1, \dots, y_s) = y^\alpha x - P(y_1, \dots, y_s)$$

and

$$Q_i(x, y_{i+1}, \dots, y_s) = \text{res}_{y_i}(Q_{i-1}(x, y_i, \dots, y_s), y_i^{n_i} - 1) \in \bar{K}[x, y_{i+1}, \dots, y_s], \quad i = 1, \dots, s.$$

In detail

$$\text{res}_{\bar{n}}(\Omega) = \text{res}_{y_s}(\dots \text{res}_{y_1}(y_1^{\alpha_1} \dots y_s^{\alpha_s} x - P(y_1, \dots, y_s), y_1^{n_1} - 1), \dots, y_s^{n_s} - 1).$$

Clearly $\lambda = \Omega(\xi) = P(\xi)/\xi^\alpha$ for some $\xi \in \mu_{\bar{n}}$ if and only if $\text{res}_{\bar{n}}(\Omega)(\lambda) = 0$.

Given a lattice Λ of rank s , a basis \mathcal{V} of Λ and a multi-index $\bar{n} \in \mathbb{N}_{\text{co}(p)}^s$, we let as before $\Lambda_{\bar{n},\mathcal{V}}$ denote the product sublattice $\bigoplus n_i \mathbb{Z} v_i$ of Λ . In the p -free case we deduce from Lemma 2.9 the following expression for the Chebyshev-Dickson polynomial $\text{CharPoly}_{a,\bar{n},\mathcal{V}}$ in (18) as the multivariate iterated resultant of the symbol $\sigma_{a,\mathcal{V}}$ in (15). We leave the details to the reader.

Proposition 2.11. *In the notation as above, the characteristic polynomial of the restriction $\Delta_a | \Lambda_{\bar{n},\mathcal{V}}$, where $a \in \mathcal{F}(\Lambda, \bar{K})$ and $\bar{n} \in \mathbb{N}_{\text{co}(p)}^s$, can be expressed as follows:*

$$\text{CharPoly}_{a,\bar{n},\mathcal{V}} = \text{res}_{\bar{n}}(\sigma_{a,\mathcal{V}}).$$

Cf. [HMP, 3.3] for an alternative expression of the characteristic polynomials of σ^+ -automata on multi-dimensional grids in terms of iterated resultants.

Example 2.12. Using the above proposition we derive the following expression for the classical Chebyshev-Dickson polynomials T_n of the first kind:

$$T_n(x) = \text{res}_y(xy + y^2 + 1, y^n + 1).$$

Remark 2.13. Despite the fact that any Chebyshev-Dickson system satisfies the divisibility property, its individual members can be arbitrary polynomials. Let us show for instance that given a degree $d > 0$ polynomial $P \in \bar{K}[x]$, where $K = \text{GF}(q)$, there exists a function $a \in \mathcal{F}^0(\mathbb{Z}, \bar{K})$ such that $P = \text{CharPoly}_{a,d,e_1}$. Indeed enumerating arbitrarily the roots $z_1, \dots, z_d \in \bar{K}$ of P we consider a function \hat{a}_* on G^\vee with $\hat{a}_*(i) = z_{i+1}$, $i = 0, \dots, d-1$, where $G = \mathbb{Z}/d\mathbb{Z}$. Letting $a_* = F^{-1}(\hat{a}_*) \in \mathcal{F}(G, \bar{K})$ we push a_* backward via $\mathbb{Z} \twoheadrightarrow G$ to a function $a \in \mathcal{F}^0(\mathbb{Z}, \bar{K})$ supported on the interval $[0, \dots, d-1]$. Then a is as required.

3. COUNTING POINTS ON SYMBOLIC VARIETY

3.1. Harmonic characters as points on symbolic variety. We let as before $K = \text{GF}(p^r)$. Given $\bar{a} = (a_1, \dots, a_t) \in (\mathcal{F}^0(\Lambda, \bar{K}))^t$, we establish in Proposition 3.1 below a natural bijection between the set of points on the symbolic variety $\Sigma_{\bar{a}, \mathcal{V}}$ in (17) and the set $\text{Char}_{\bar{a}\text{-harm}}(\Lambda, \bar{K}^\times)$ of all \bar{a} -harmonic characters of Λ .

Since \bar{K}^\times is a torsion group, given a basis $\mathcal{V} = (v_1, \dots, v_s)$ of Λ , any character $g^\vee \in \text{Char}(\Lambda, \bar{K}^\times)$ is (\bar{n}, \mathcal{V}) -periodic for $\bar{n} = (n_1, \dots, n_s)$, where $n_i = \text{ord}(g^\vee(v_i)) \in \mathbb{N}_{\text{co}(p)}$, $i = 1, \dots, s$. Letting $G = G_{\bar{n}, \mathcal{V}} = \Lambda / \Lambda_{\bar{n}, \mathcal{V}}$ we have $g^\vee = h^\vee \circ \pi$ for a character $h^\vee \in G^\vee$, where $\pi : \Lambda \twoheadrightarrow G$. By virtue of the Pushforward Lemma 1.1, g^\vee is \bar{a} -harmonic if and only if h^\vee is \bar{a}_* -harmonic. Consequently

$$\text{Char}_{\bar{a}\text{-harm}}(\Lambda, \bar{K}^\times) = \bigcup_{\bar{n} \in \mathbb{N}_{\text{co}(p)}^s} (G_{\bar{n}, \mathcal{V}})_{\bar{a}_*\text{-harm}}^\vee.$$

For any $\bar{a} = (a_1, \dots, a_t) \in (\mathcal{F}^0(\Lambda, \bar{K}))^t$ and $\bar{n} = (n_1, \dots, n_s) \in \mathbb{N}_{\text{co}(p)}^s$, we consider the following over-determined system of algebraic equations, cf. (15):

$$(19) \quad \sigma_{a_j, \mathcal{V}}(x_1, \dots, x_s) = 0, \quad x_i^{n_i} = 1, \quad i = 1, \dots, s, \quad j = 1, \dots, t.$$

We let $\Sigma_{\bar{a}, \bar{n}, \mathcal{V}} = \Sigma_{\bar{a}, \mathcal{V}} \cap \mu_{\bar{n}}$ denote the set of all solutions of (19), or in other words the set of all points on the symbolic variety $\Sigma_{\bar{a}, \mathcal{V}}$ in (15) whose multi-torsion orders divide $\bar{n} = (n_1, \dots, n_s)$. The following result yields (b) of Theorem 0.2 in the Introduction.

Proposition 3.1. *Given a basis \mathcal{V} of Λ , the natural bijection*

$$\text{Char}(\Lambda, \bar{K}^\times) \xrightarrow{\cong} (\bar{K}^\times)^s$$

restricts to

$$\text{Char}_{\bar{a}\text{-harm}}(\Lambda, \bar{K}^\times) \xrightarrow{\cong} \Sigma_{\bar{a}, \mathcal{V}}$$

and further yields the bijections

$$(20) \quad \Sigma_{\bar{a}, \bar{n}, \mathcal{V}} = \Sigma_{\bar{a}, \mathcal{V}} \cap \mu_{\bar{n}} \xrightarrow{\cong} (G_{\bar{n}, \mathcal{V}})_{\bar{a}_*\text{-harm}}^\vee \xrightarrow{\cong} V(\bar{a}_*),$$

where $\bar{a}_ = \pi_* \bar{a}$ is the pushforward of \bar{a} under the canonical surjection $\pi : \Lambda \twoheadrightarrow G_{\bar{n}, \mathcal{V}} = \Lambda / \Lambda_{\bar{n}, \mathcal{V}}$. Consequently*

$$d(\bar{a}, \Lambda_{\bar{n}, \mathcal{V}}) = \text{card } V(\bar{a}_*) = \text{card } \Sigma_{\bar{a}, \bar{n}, \mathcal{V}}.$$

Proof. For a character $h^\vee \in G_{\bar{n}, \mathcal{V}}^\vee$, letting $g^\vee = h^\vee \circ \pi \in \text{Char}(\Lambda, \bar{K}^\times)$ and $\xi_i = g^\vee(v_i) \in \bar{K}^\times$ we obtain $\xi_i^{n_i} = 1 \ \forall i = 1, \dots, s$ (indeed $n_i v_i \in \Lambda_{\bar{n}, \mathcal{V}} \ \forall i$). Moreover $h^\vee \in (G_{\bar{n}, \mathcal{V}})_{\bar{a}_*}^\vee$ if and only if $\forall j = 1, \dots, t$,

$$\begin{aligned} h^\vee * (a_j)_* = 0 &\iff g^\vee * a_j = 0 \iff g^\vee * \left(\sum_{v \in L} a_j(v) \delta_v \right) = 0 \\ &\iff \sum_{v = \sum_{i=1}^s \alpha_i v_i \in L} a_j(v) (g^\vee)^{-1}(v) = 0 \iff \sigma_{a_j, \mathcal{V}}(\xi_1, \dots, \xi_s) = 0, \end{aligned}$$

and so $\xi = (\xi_1, \dots, \xi_s) \in \bigcap_{j=1}^t \Sigma_{a_j, \mathcal{V}} = \Sigma_{\bar{a}, \mathcal{V}}$. Vice versa, given a solution $\xi = (\xi_1, \dots, \xi_s) \in (\bar{K}^\times)^s$ of (19), letting $g^\vee(v_i) = \xi_i$ defines an (\bar{n}, \mathcal{V}) -periodic character $g^\vee \in \text{Char}(\Lambda, \bar{K}^\times)$. By the same argument as above, g^\vee and the pushforward character $h^\vee = \pi_*(g^\vee) \in (G_{\bar{n}, \mathcal{V}})^\vee$ are \bar{a} - and \bar{a}_* -harmonic, respectively. The correspondence $\xi = (\xi_1, \dots, \xi_s) \longleftrightarrow h^\vee$ yields the first bijection in (20). As for the second one, see 2.2(b). \square

3.2. Criteria of harmonicity. We let as before $K = \text{GF}(p)$. Given a basis \mathcal{V} of Λ , a sequence $\bar{a} \in (\mathcal{F}^0(\Lambda, \bar{K}))^t$ and a multi-index $\bar{n} \in \mathbb{N}_{\text{co}(p)}^s$, we let $G = \Lambda / \Lambda_{\bar{n}, \mathcal{V}}$ and $\bar{a}_* = \pi_* \bar{a}$, where $\pi : \Lambda \twoheadrightarrow G$. We fix a minimal $q_0 = q(\bar{a}_*) = p^{r_0}$ ($r_0 > 0$) such that $\widehat{a_j}(G^\vee) \subseteq \text{GF}(q_0) \ \forall j = 1, \dots, t$. The preceding results lead to the following criteria.

Theorem 3.2. (a) *With the notation as above, the following conditions are equivalent.*

- (i) *There exists a nonzero (\bar{n}, \mathcal{V}) -periodic³ \bar{a} -harmonic function on Λ .*
- (ii) $V(\bar{a}_*) := \bigcap_{j=1}^t V(a_{j*}) \neq \emptyset$.
- (iii) *The system (19) has a solution $\xi = (\xi_1, \dots, \xi_s) \in \Sigma_{\bar{a}, \bar{n}, \mathcal{V}} = \Sigma_{\bar{a}, \mathcal{V}} \cap \mu_{\bar{n}}$.*
- (b) *Furthermore $\ker(\Delta_{\bar{a}_*}) \subseteq (\mathcal{F}(G, \bar{K}), *)$ coincides with the principal convolution ideal generated by the function $\prod_{j=1}^t \left(\delta_e - \Delta_{(a_j)_*}^{q_0-1}(\delta_e) \right)$, and*

$$\prod_{j=1}^t \left(1_G - \Delta_{(a_j)_*}^{q_0-1} \right) : \mathcal{F}(G, \bar{K}) \twoheadrightarrow \ker(\Delta_{\bar{a}_*})$$

is an orthogonal projection.

- (c) *For $t = 1$ and $a_1 = a$, (i)-(iii) are equivalent to every one of the following conditions:*
- (iv) $(\Delta_{a_*})^{q_0-1}(\delta_e) \neq \delta_e$ or, equivalently, $(\Delta_{a_*})^{q_0-1} \neq 1_G$.
- (v) *The sequence $(\Delta_{a_*}^k(\delta_e))_{k \geq 0} \subseteq \mathcal{F}(G, \bar{K})$ is not periodic.*

Proof. The equivalences (i) \iff (ii) \iff (iii) follow immediately from 2.2 and 3.1. Since the function $\widehat{(a_j)_*} \in \mathcal{F}(G^\vee, \bar{K})$ takes values in the field $\text{GF}(q_0)$ we have $\delta_{V(\widehat{(a_j)_*})} = 1 - \widehat{(a_j)_*}^{q_0-1}$. For every $j = 1, \dots, t$ the Fourier transform sends $1_G - \Delta_{(a_j)_*}^{q_0-1}$ to the operator of multiplication by $\delta_{V((a_j)_*)}$ in $\mathcal{F}(G^\vee, \bar{K})$, which coincides with the orthogonal projection onto the corresponding principal ideal. This yields (b) and (c). \square

³I.e., stable under the shifts by elements of $\Lambda_{\bar{n}, \mathcal{V}}$.

Remarks 3.3. 1. For $t = 1$ and $a = a_1$ we have $\widehat{a}_*^{q_0} = \widehat{a}_*$ and so $\Delta_{a_*}^{q_0+1}(\delta_e) = \Delta_{a_*}(\delta_e) = a_*$. Consequently the truncated sequence $(\Delta_{a_*}^k(\delta_e))_{k \geq 1}$ (which starts with a_*) is periodic with period l dividing $q_0 - 1$. Whereas the sequence in (iv) (which starts with δ_e) is periodic if and only if Λ does not admit a nonzero a -harmonic (\bar{n}, \mathcal{V}) -periodic function. In the latter case Δ_{a_*} is invertible of finite order in the group $\text{Aut}(\Lambda/\Lambda_{\bar{n}, \mathcal{V}}, \bar{K})$.

2. For $K = \text{GF}(2)$ and $G = \mathbb{Z}/n\mathbb{Z}$, according to [J, 1.1.7] or [MOW], we have

$$K[G]^\times = (K[x]/(x^n - 1))^\times \cong \mathbb{Z}/\nu\mathbb{Z},$$

where

$$\nu = \nu(n) = 2^n \prod_{d|n} \left(1 - \frac{1}{2^{f(d)}}\right)^{g(d)}$$

with $f(n) = \text{ord}_n(2) = \min\{j : 2^j \equiv 1 \pmod{n}\}$ and $g(n) = \varphi(n)/f(n)$. Here φ stands for the Euler totient function. We recall that G admits a nonzero a^+ -harmonic function if and only if $n \equiv 0 \pmod{3}$. Otherwise the minimal period l as in (1) above coincides with the order of \tilde{a}^+ in the cyclic group $\mathbb{Z}/\nu\mathbb{Z}$, so $l \mid \nu$.

3. For $K = \text{GF}(2)$, $t = 1$ and $a = a^+$, $\Delta_{a_*}^{q_0-1} : \mathcal{F}(G, \bar{K}) \rightarrow (\ker(\Delta_{a_*}))^\perp$ is the orthogonal projection onto the space $(\ker(\Delta_{a_*}))^\perp$ of all winning patterns of the 'Lights Out' game on the toric grid $G = \mathbb{Z}_{\bar{n}}$; see [Za1, §2.8] or §2 in the Introduction.

4. CONVOLUTION EQUATIONS OVER FINITE FIELDS

We fix a Galois field $K = \text{GF}(q)$ with $q = p^r$ and a finite abelian group G of order coprime with p . Let $\Delta_{\bar{a}} = (\Delta_{a_1}, \dots, \Delta_{a_t})$ be a system of convolution operators with kernel $\bar{a} \in (\mathcal{F}(G, K))^t$. Clearly the dimension of the space of solutions $\ker(\Delta_{\bar{a}})$ is the same in $\mathcal{F}(G, K)$ and in $\mathcal{F}(G, \bar{K})$. We show in Theorem 4.4 below that, moreover, the former subspace can be recovered by taking traces of \bar{a} -harmonic characters (with values in \bar{K}).

We let $\phi_q : \xi \mapsto \xi^q$ denote the Frobenius automorphism of $\bar{K} = \overline{\text{GF}(q)}$. By the same letter we denote the induced action $\phi_q : f \mapsto f^{\phi_q} = f^q$ on the function spaces $\mathcal{F}(G, \bar{K})$ and $\mathcal{F}(G^\vee, \bar{K})$, respectively.

The rings of invariants

$$[\mathcal{F}(G, \bar{K})]^{\phi_q} = \mathcal{F}(G, K) \quad \text{and} \quad [\mathcal{F}(G^\vee, \bar{K})]^{\phi_q} = \mathcal{F}(G^\vee, K)$$

do not correspond to each other under the Fourier transform. The restriction $D_q = \phi_q|_{G^\vee}$ to the image of $G^\vee \hookrightarrow \mathcal{F}(G, \bar{K})$ is just the multiplication by q in the abelian group G^\vee . We keep again the same symbol D_q for the induced automorphism of the function space $\mathcal{F}(G^\vee, \bar{K})$. The latter one being different from ϕ_q , we let α_q denote the automorphism $\phi_q \circ (D_q)^{-1}$ of $\mathcal{F}(G^\vee, \bar{K})$ measuring this difference. In the next simple lemma (cf. [J]) we show that α_q is the Fourier dual of the Frobenius automorphism acting on $\mathcal{F}(G, \bar{K})$.

Lemma 4.1. *The automorphism $\alpha_q \in \text{Aut}(\mathcal{F}(G^\vee, \bar{K}))$ is the Fourier dual of $\phi_q \in \text{Aut}(\mathcal{F}(G, \bar{K}))$. Hence the Fourier image $F(\mathcal{F}(G, K))$ coincides with the subalgebra $(\mathcal{F}(G^\vee, \bar{K}))^{\alpha_q} \subseteq \mathcal{F}(G^\vee, \bar{K})$ of α_q -invariants.*

Proof. For any $f \in \mathcal{F}(G, \bar{K})$ we have

$$\left(\widehat{f}\right)^{\phi_q} = \widehat{f^{\phi_q}} \circ D_q.$$

Indeed, for any $g^\vee \in G^\vee$,

$$\left(\hat{f}(g^\vee)\right)^q = \left(\sum_{v \in G} f(v)g^\vee(v)\right)^q = \sum_{v \in G} f^{\phi_q}(v)(g^\vee)^{\phi_q}(v) = \widehat{f^{\phi_q}}((g^\vee)^{\phi_q}).$$

Therefore

$$f \in \mathcal{F}(G, K) \iff f = f^{\phi_q} \iff \hat{f} = \widehat{f^{\phi_q}} \iff \hat{f} \circ D_q = (\hat{f})^{\phi_q} \iff \hat{f} = \alpha(\hat{f}),$$

as stated. \square

Now one can easily deduce the following fact.

Corollary 4.2. *For any $\bar{a} \in (\mathcal{F}(G, K))^t$, the locus $V(\bar{a}) \subset G^\vee$ of \bar{a} -harmonic characters is D_q -stable.*

This leads to a direct sum decomposition of the space of solutions, see 4.3(b) below. For a function $f \in \mathcal{F}(G, \bar{K})$ we let $GF(q(f))$, where $q(f) = q^{r(f)}$, denote the minimal subfield of \bar{K} generated by K and by the image $f(G)$. The trace of f is

$$\text{Tr}(f) = \text{Tr}_{GF(q(f)):GF(q)}(f) = f + f^q + \dots + f^{q^{r(f)-1}} \in \mathcal{F}(G, K).$$

Proposition 4.3. *For any $\bar{a} \in (\mathcal{F}(G, K))^t$ the following hold.*

- (a) *There is a bijection between the set of traces $\text{Tr}(g^\vee) \in \mathcal{F}(G, K)$ of all \bar{a} -harmonic characters $g^\vee \in V(\bar{a})$ and the orbit space $V(\bar{a})/\langle D_q \rangle$ of the cyclic group $\langle D_q \rangle$ acting on $V(\bar{a})$.*
- (b) *Given a set of representatives $g_1^\vee, \dots, g_m^\vee$ of the $\langle D_q \rangle$ -orbits on $V(\bar{a})$, there is a decomposition into orthogonal direct sum of convolution ideals*

$$\ker(\Delta_{\bar{a}}) = \bigoplus_{i=1}^m (\text{Tr}(g_i^\vee)) \subseteq \mathcal{F}(G, K).$$

- (c) *For any $g^\vee \in G^\vee$ one has*

$$g^\vee = \frac{1}{\text{ord}(G)} \sum_{g \in G} g^\vee(g^{-1})h_g, \quad \text{where } h_g(x) = \text{Tr}(g^\vee(gx)) = \tau_g(\text{Tr}(g^\vee(x))).$$

Proof. By virtue of 4.2 for any character $g^\vee \in V(\bar{a})$ we have

$$h = \text{Tr}(g^\vee) = g^\vee + (g^\vee)^q + \dots + (g^\vee)^{q^{r(g^\vee)-1}} \in \ker(\Delta_{\bar{a}}) \cap \mathcal{F}(G, K).$$

Letting

$$O(g^\vee) = \{g^\vee, (g^\vee)^q, \dots, (g^\vee)^{q^{r(g^\vee)-1}}\}$$

be the orbit of g^\vee under the action of the cyclic group $\langle D_q \rangle$ on $V(\bar{a})$, one can easily deduce that $\text{card}(O(g^\vee)) = r(g^\vee)$ and, by (12),

$$\hat{h} = \text{ord}(G) \sum_{i=0}^{r(g^\vee)-1} \delta_{(g^\vee)^{-q^i}} = \text{ord}(G) \delta_{O((g^\vee)^{-1})}.$$

Now $h \iff O((g^\vee)^{-1})$ is the correspondence required in (a). In turn (a) implies (b). Whereas (c) follows by using the orthogonality relations for characters. Indeed by

virtue of (12), for any $x \in G$ one has

$$\begin{aligned}
\sum_{g \in G} g^\vee(g^{-1})h_g(x) &= \sum_{g \in G} g^\vee(g^{-1}) \sum_{i=0}^{r(g^\vee)-1} (g^\vee)^{q^i}(gx) \\
&= \sum_{i=0}^{r(g^\vee)-1} \left(\sum_{g \in G} (g^\vee)^{-q^i}(g^{-1})g^\vee(g^{-1}) \right) (g^\vee)^{q^i}(x) = \sum_{i=0}^{r(g^\vee)-1} \widehat{(g^\vee)^{-q^i}}(g^\vee)(g^\vee)^{q^i}(x) \\
&= \text{ord}(G) \sum_{i=0}^{r(g^\vee)-1} \delta_{(g^\vee)^{q^i}}(g^\vee)(g^\vee)^{q^i}(x) = \text{ord}(G)g^\vee(x).
\end{aligned}$$

□

The following result is straightforward from 3.1 and 4.3(c).

Theorem 4.4. (a) *For a finite abelian group G of order coprime to p and for any $\bar{a} \in (\mathcal{F}(G, K))^t$, where $K = \text{GF}(p^r)$, the kernel $\ker(\Delta_{\bar{a}})$ is spanned over K by the shifts of traces of \bar{a} -harmonic characters $g^\vee \in G_{\bar{a}\text{-harm}}^\vee$.*
(b) *Similarly, for any sublattice $\Lambda' \subseteq \Lambda$ of finite index coprime to p and for any $\bar{a} \in (\mathcal{F}(\Lambda, K))^t$, the kernel $\ker(\Delta_{\bar{a}}|_{\mathcal{F}_{\Lambda'}(\Lambda, K)})$ is spanned over K by the shifts of traces of \bar{a} -harmonic characters $g^\vee \in \text{Char}_{\bar{a}\text{-harm}}(\Lambda, \bar{K}^\times)$ with $\Lambda' \subseteq \ker(g^\vee)$.*

5. CHARACTERISTIC SUBLATTICES AND TRANSLATION INVARIANT SUBSPACES

5.1. Lattice characters with values in \bar{K}^\times . Given a lattice Λ , a character of Λ with values in \bar{K}^\times is just a homomorphism $g^\vee : \Lambda \rightarrow \bar{K}^\times$. It is called \bar{a} -harmonic, where $\bar{a} = (a_1, \dots, a_t) \in (\mathcal{F}^0(\Lambda, \bar{K}))^t$, if $\Delta_{a_j}(g^\vee) = 0 \forall j = 1, \dots, t$. We let as before $\text{Char}(\Lambda, \bar{K}^\times)$ denote the set of all characters on Λ with values in \bar{K}^\times and $\text{Char}_{\bar{a}\text{-harm}}(\Lambda, \bar{K}^\times)$ the set of all \bar{a} -harmonic characters.

Example 5.1. For $K = \text{GF}(2)$, $\Lambda = \mathbb{Z}^2$, $t = 1$, $a_1 = a^+$ and for a primitive cubic root of unity $\omega \in \mu_3$,

$$\theta = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 1 & \omega & \omega^2 & 1 & \omega & \dots \\ \dots & 1 & \omega & \omega^2 & 1 & \omega & \dots \\ \dots & 1 & \omega & \omega^2 & 1 & \omega & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

is an a^+ -harmonic character with values in $\mu_3 \subseteq \bar{K}^\times$.

Remarks 5.2. 1. Clearly $\Lambda / \ker(\theta) \cong \mathbb{Z}/m\mathbb{Z}$ for a character $\theta \in \text{Char}(\Lambda, \bar{K}^\times)$ if and only if $\text{ord}(\theta) = m$. Vice versa given a sublattice $\Lambda' \subseteq \Lambda$ with $\Lambda/\Lambda' \cong \mathbb{Z}/m\mathbb{Z}$, where $m \in \mathbb{N}_{\text{co}(p)}$, we have $\Lambda' = \ker(\theta)$ for the character θ on Λ defined via

$$\theta : \Lambda \twoheadrightarrow \Lambda/\Lambda' \xrightarrow{\cong} \mathbb{Z}/m\mathbb{Z} \xrightarrow{\cong} \mu_m \hookrightarrow \bar{K}^\times.$$

2. We note that $\Lambda(\theta') = \varepsilon(\Lambda(\theta))$ for two characters $\theta, \theta' \in \text{Char}(\mathbb{Z}^s, \bar{K}^\times)$ and for some $\varepsilon \in \mathbf{SL}(s, \mathbb{Z})$ if and only if $\text{ord}(\theta') = \text{ord}(\theta)$. In the latter case $\theta' = \delta \circ \theta \circ \varepsilon$, where $\delta \in \text{Aut}(\mathbb{Z}/m\mathbb{Z}) \cong (\mathbb{Z}/m\mathbb{Z})^\times$ with $m = \text{ord}(\theta)$.

Fixing a basis $\mathcal{V} = (v_1, \dots, v_s)$ of Λ , for a lattice vector $v = \sum_{i=1}^s \alpha_i(v) v_i \in \Lambda$ we let $\vec{v} = (\alpha_1, \dots, \alpha_s) \in \mathbb{Z}^s$, where $\alpha_i = \alpha_i(v)$. Below $\langle \cdot, \cdot \rangle$ stands for the standard bilinear form on \mathbb{Z}^s . We observe the following.

Proposition 5.3. (a) *Given a primitive m th root of unity $\zeta \in \mu_m$ ($m \in \mathbb{N}_{\text{co}(p)}$) and a primitive lattice vector $v_0 \in \Lambda$, the formula*

$$(21) \quad \theta(v) = \zeta^{\langle \vec{v}, \vec{v}_0 \rangle}$$

defines a character $\theta \in \text{Char}(\Lambda, \bar{K}^\times)$ of order m .

(b) *Vice versa any character $\theta \in \text{Char}(\Lambda, \bar{K}^\times)$ can be written as in (21) for a suitable primitive m th root of unity $\zeta \in \mu_m$ of order $m = \text{ord}(\theta) \in \mathbb{N}_{\text{co}(p)}$ and a suitable primitive lattice vector $v_0 = v_\theta = \sum_{i=1}^s \alpha_i(v_\theta) v_i \in \Lambda$ with $\alpha_i(v_\theta) \in \{0, \dots, m-1\}$.*

(c) *Consequently the period lattice of θ :*

$$\Lambda(\theta) = \ker(\theta) = \{v \in \Lambda : \langle \vec{v}, \vec{v}_\theta \rangle \equiv 0 \pmod{m}\}$$

is an index m sublattice of Λ .

(d) *The character θ as in (21) is a -harmonic, where $a \in \mathcal{F}^0(\Lambda, \bar{K})$, if and only if*

$$(22) \quad (\theta * a)(0) = \sigma_{a, \mathcal{V}}(\xi) = 0 \quad \text{that is} \quad \xi \in \Sigma_{a, \mathcal{V}},$$

where $\xi = (\zeta^{\alpha_1(v_\theta)}, \dots, \zeta^{\alpha_s(v_\theta)})$ and $\sigma_{a, \mathcal{V}}$ is the symbol of Δ_a w.r.t. the basis \mathcal{V} .

Proof. The proof of (a) is straightforward. To show the converse we let $\xi_i = \theta(e_i) \in \bar{K}^\times$ and $n_i = \text{ord}(\xi_i)$, $i = 1, \dots, s$. We let also $m = m(\theta) = \text{lcm}(n_1, \dots, n_s)$ be the exponent of the group $\mu_{\bar{n}}$. For a primitive m th root of unity $\tilde{\zeta} \in \mu_m$ we write $\xi_i = \tilde{\zeta}^{b_i}$, where $n_i = m / \gcd(b_i, m)$. Letting further $d = \gcd(b_1, \dots, b_s)$, $b_i = d\alpha_i$ and $d_i = \gcd(b_i, m)$ we obtain $\gcd(d_1, \dots, d_s) = 1$ and so $\gcd(d, m) = 1$. Hence $\zeta = \tilde{\zeta}^d$ is again a primitive m th root of unity and $\xi_i = \zeta^{\alpha_i}$, $i = 1, \dots, s$. Therefore $\theta : v \mapsto \zeta^{\langle \vec{v}, \vec{v}_\theta \rangle}$ for a primitive vector $v_\theta = \sum_{i=1}^s \alpha_i(v_\theta) v_i \in \Lambda$, where $\alpha_i \in \{0, \dots, m-1\}$, $i = 1, \dots, s$.

Now (22) follows from the equalities

$$\begin{aligned} \theta * a(0) &= \sum_{u \in \Lambda} \theta(u) a(-u) = \sum_{u \in \Lambda} a(u) \zeta^{-\langle \vec{u}, \vec{v}_\theta \rangle} \\ &= \sum_{u \in L} a(u) \zeta^{-\alpha_1(u) \alpha_1(v_\theta)} \cdot \dots \cdot \zeta^{-\alpha_s(u) \alpha_s(v_\theta)} = \sigma_{a, \mathcal{V}}(\xi). \end{aligned}$$

□

Example 5.4. A character $\theta \in \text{Char}(\mathbb{Z}^s, \bar{K}^\times)$, $v \mapsto \zeta^{\langle v, v_0 \rangle}$ as in (21), where $K = \text{GF}(2)$, is a^+ -harmonic if and only if

$$(23) \quad (\theta * a^+)(0) = 1 + \sum_{i=1}^s (\zeta^{\alpha_i(v_0)} + \zeta^{-\alpha_i(v_0)}) = 0.$$

Remark 5.5. A primitive lattice vector $v_0 \in \Lambda$ will be called a -exceptional if the symbol $\sigma_{a_*} \in \bar{K}[x, x^{-1}]$ is a Laurent monomial, where $a_* = \pi_* a \in \mathcal{F}(\mathbb{Z}, \bar{K})$ for the surjection $\pi : \Lambda \rightarrow \mathbb{Z}$, $v \mapsto \langle \vec{v}, \vec{v}_0 \rangle$. Clearly there exist many non a -exceptional vectors $v_0 \in \Lambda$ as soon as $\text{card}(\text{supp}(a)) \geq 2$. For such a vector v_0 any nonzero root ζ of the Laurent polynomial σ_{a_*} gives rise to an a -harmonic character $\theta \in \text{Char}(\Lambda, \bar{K}^\times)$, $v \mapsto \zeta^{\langle \vec{v}, \vec{v}_0 \rangle}$.

5.2. Periodicity of solutions of convolution equations. A general convolution equation on a lattice Λ of rank ≥ 2 does admit aperiodic solutions. For instance on $\Lambda = \mathbb{Z}^2$ there are aperiodic a^+ -harmonic functions with values in $K = \text{GF}(2)$. Indeed consider the strip $S = \mathbb{Z} \times \{0, -1\} \subseteq \Lambda$ of width 2. Any function $f_0 : S \rightarrow \bar{K}$ admits a unique a^+ -harmonic extension $f_0 \rightsquigarrow f$ to Λ given via

$$f(m, n) = f_0(m, -1) + f_0(m, 0) + f_0(m + n, 0) + f_0(m - n, 0), \quad m > 0,$$

on the upper halfplane and symmetrically on the lower one. Clearly for a generic f_0 this extension f is aperiodic that is $\Lambda(f) = \{0\}$. The space of all such a^+ -harmonic functions is of infinite dimension.

Furthermore there are bi-periodic a^+ -harmonic functions on Λ with arbitrarily large pairs of periods, see [Za₁].

However all solutions of convolution equations on rank 1 lattices are periodic with a period depending only on the equation. Indeed the following holds.

Lemma 5.6. *For any $a \in \mathcal{F}^0(\mathbb{Z}, \bar{K}) \setminus \{0\}$, every a -harmonic function $f \in \ker(\Delta_a)$ is m_a -periodic for a certain $m_a > 0$ depending only on a . Consequently the subspace*

$$\ker(\Delta_a) \subseteq \mathcal{F}_{\Lambda'}(\Lambda, \bar{K}), \quad \text{where } \Lambda' = m_a \mathbb{Z} \subseteq \Lambda = \mathbb{Z},$$

is of finite dimension.

Proof. Replacing a by $a * \delta_n$ with a suitable n we may suppose that $a = \sum_{i=0}^N a(i) \delta_{-i}$ with $N \geq 0$ and $a(0), a(N) \neq 0$. For any $f \in \ker(\Delta_a)$ we have

$$0 = f * a(0) = a(0)f(0) + a(1)f(1) + \dots + a(N)f(N).$$

Hence

$$f(N) = b_0 f(0) + \dots + b_{N-1} f(N-1),$$

where $b_i = -\frac{a(i)}{a(N)}$ and so $b_0 \neq 0$. Therefore the linear transformation

$$\varphi : \mathbb{A}_{\bar{K}}^N \rightarrow \mathbb{A}_{\bar{K}}^N, \quad (f_0, \dots, f_{N-1}) \mapsto (f_1, \dots, f_N)$$

with $\det(\varphi) = \pm b_0$ is invertible, hence of finite order, say, m_a . This shows that f is periodic of period m_a . \square

For instance any a^+ -harmonic function on \mathbb{Z} is periodic of period 3. Moreover any a^+ -harmonic function on $\Lambda = \mathbb{Z}^2$ which is periodic on the strip $S = \mathbb{Z} \times \{0, -1\}$ is also bi-periodic. In the same fashion, one can easily prove the following fact.

Corollary 5.7. *Let $\bar{a} = (a_1, \dots, a_l) \in (\mathcal{F}^0(\Lambda, \bar{K}))^l$ and suppose that the convolution ideal $(a_1, \dots, a_l) \subseteq \mathcal{F}^0(\Lambda, \bar{K})$ contains $s = \text{rk}(\Lambda)$ nonzero functions $b_1, \dots, b_s \in \mathcal{F}^0(\Lambda, \bar{K})$ such that $\text{supp}(b_j) \subseteq \mathbb{Z}v_j$ ($j = 1, \dots, s$), where $v_1, \dots, v_s \in \Lambda$ are linearly independent. Then any function $f \in \ker(\Delta_{\bar{a}}) = \bigcap_{j=1}^l \ker(\Delta_{a_j})$ is pluri-periodic and its period lattice $\Lambda(f)$ contains a rank s sublattice $\Lambda' = \sum_{j=1}^s m_j \mathbb{Z}v_j$, where $m_j > 0$, $j = 1, \dots, s$, depend only on \bar{a} .*

5.3. Translation invariant subspaces. The following simple observation will be useful. For any finitely generated abelian group G with a Sylow p -subgroup $G(p)$ one has

$$G(p) = \bigcap_{\theta \in \text{Char}(G, \bar{K}^\times)} \ker(\theta) \quad \text{and} \quad \text{Char}(G, \bar{K}^\times) = \pi^* \text{Char}(G/G(p), \bar{K}^\times),$$

where $\pi : G \twoheadrightarrow G/G(p)$.

Let us introduce the following notions.

Definition 5.8. A subspace $E \subset \mathcal{F}(\Lambda, \bar{K})$ spanned by a finite set of characters $\theta_1, \dots, \theta_m \in \text{Char}(\Lambda, \bar{K}^\times)$ will be called characteristic. We note that any characteristic subspace is translation invariant.

A sublattice will be called characteristic if it is of the form

$$\Lambda' = \bigcap_{i=1}^m \ker(\theta_i),$$

where $\theta_1, \dots, \theta_m \in \text{Char}(\Lambda, \bar{K}^\times)$.

Lemma 5.9. A sublattice $\Lambda' \in \mathcal{L}$ is characteristic if and only if $\Lambda' \in \mathcal{L}^0$.

Proof. If $\Lambda' \subseteq \Lambda$ is characteristic then $\Lambda' = \ker(\varphi)$, where $\varphi = (\theta_1, \dots, \theta_m) : \Lambda \rightarrow \bigoplus_{i=1}^m \mu_{n_i}$ with $n_i = \text{ord}(\theta_i) \in \mathbb{N}_{\text{co}(p)}$, $i = 1, \dots, m$. Hence the index of Λ' in Λ is coprime with p i.e., $\Lambda' \in \mathcal{L}^0$.

Conversely assuming that $\Lambda' \in \mathcal{L}^0$, the order of the group $G = \Lambda/\Lambda'$ is coprime with p . Hence G is a product of cyclic groups μ_{m_i} of orders $m_i \in \mathbb{N}_{\text{co}(p)}$, $i = 1, \dots, m$. The compositions

$$\Lambda \twoheadrightarrow G \twoheadrightarrow \mu_{m_i}, \quad i = 1, \dots, m,$$

yield characters $\theta_i \in \text{Char}(\Lambda, \bar{K}^\times)$ with $\Lambda' = \bigcap_{i=1}^m \ker(\theta_i)$, and so the sublattice $\Lambda' \subseteq \Lambda$ is characteristic. \square

Remark 5.10. Given a finite group G of order coprime with p and a function $f \in \mathcal{F}(G, \bar{K})$, the subgroup of periods $\Lambda(f) \subseteq G$ is a characteristic subgroup. It can be recovered by $\text{supp}(\hat{f})$ as follows:

$$\Lambda(f) = \bigcap_{g^\vee \in \text{supp}(\hat{f})} \ker(g^\vee).$$

Indeed for $g \in G$, $f_g = f \iff \hat{f} \cdot g = \hat{f} \iff g|_{\text{supp}(\hat{f})} = 1 \iff g \in \bigcap_{g^\vee \in \text{supp}(\hat{f})} \ker(g^\vee)$.

Let $E \subseteq \mathcal{F}(\Lambda, \bar{K})$ be a translation invariant subspace of finite dimension. We notice that E consists of pluri-periodic functions. Indeed the restrictions to E of any shift has finite order. We let $G(E)$ and $\Lambda(E)$ denote the image and the kernel, respectively, of the homomorphism

$$\rho : \Lambda \rightarrow \text{Aut}(E) \cong \mathbf{GL}(n, \bar{K}), \quad v \longmapsto \tau_v.$$

Clearly $\Lambda(E)$ is the period lattice of E that is $\Lambda(E) = \bigcap_{f \in E} \Lambda(f)$. Since E is translation invariant, for any $f \in E$ one has

$$E(f) := \text{span}(\tau_v(f))_{v \in \Lambda} \subseteq E.$$

The group $G(E)$ is finite; indeed, this is a finitely generated abelian torsion group. We let $G_p(E)$ denote the Sylow p -subgroup of $G(E)$.

Proposition 5.11. *A translation invariant subspace $E \subseteq \mathcal{F}(\mathbb{Z}^s, \bar{K})$ of finite dimension is a characteristic subspace if and only if the sublattice $\Lambda(E) \subseteq \Lambda$ is⁴.*

Proof. If E is spanned by characters, say, $\theta_1, \dots, \theta_n$ then

$$\bigcap_{i=1}^n \ker(\theta_i) \subseteq \Lambda(f) \quad \forall f \in E,$$

hence $\Lambda(E) = \bigcap_{i=1}^n \ker(\theta_i)$ is a characteristic sublattice.

Conversely suppose that $\text{ind}_\Lambda(\Lambda(E)) = \text{ord}(G(E))$ is coprime with p . We have $E = \pi^*(E')$ for a suitable translation invariant subspace $E' \subseteq \mathcal{F}(G(E), \bar{K})$. By 5.8, E' is a convolution ideal spanned by characters. Hence E is spanned by characters. \square

However the period lattice of an \bar{a} -harmonic function on Λ is not necessarily characteristic, as the following example shows.

Example 5.12. (cf. [Za₁, Example 2.33]) Letting $K = \text{GF}(2)$, $t = 1$, $a_1 = a^+$ and $G = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$, the a^+ -harmonic function on G

$$h = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

lifts to $f = h \circ \pi \in \ker_{a^+}(\Lambda)$ via $\pi : \Lambda = \mathbb{Z}^2 \twoheadrightarrow G$. Thus f is a^+ -harmonic and has the period lattice $\Lambda(f) = \pi^{-1}(\Lambda(h)) = 3\mathbb{Z}e_1 + 6\mathbb{Z}e_2 \subseteq \Lambda$ of even index. By virtue of Proposition 5.11, f cannot be represented as a linear combination of characters with values in \bar{K}^\times .

Any sublattice $\Lambda' \in \mathcal{L}$ is contained in a unique minimal characteristic sublattice $\Lambda'' \in \mathcal{L}^0$, where

$$\Lambda'' = \bigcap_{\tilde{\Lambda} \in \mathcal{L}^0, \tilde{\Lambda} \supseteq \Lambda'} \tilde{\Lambda} = \bigcap_{\theta \in \text{Char}(\Lambda, \bar{K}^\times), \ker(\theta) \supseteq \Lambda'} \ker(\theta).$$

These data fit in the following commutative diagram:

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ 0 & \longrightarrow & \Lambda(E) & \longrightarrow & \Lambda & \xrightarrow{\rho} & G(E) \longrightarrow 0 \\ & & \downarrow & & \downarrow \text{id} & & \downarrow \\ 0 & \longrightarrow & \Lambda''(E) & \longrightarrow & \Lambda & \longrightarrow & G(E)/G_p(E) \longrightarrow 0 \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$

Any finite dimensional translation invariant subspace $E \subseteq \mathcal{F}(\Lambda, \bar{K})$ contains a unique maximal characteristic subspace $E^0 \subseteq E$, where

$$E^0 = \text{span} \left(\theta : \theta \in E \cap \text{Char}(\Lambda, \bar{K}^\times) \right).$$

⁴That is $\Lambda(E) \in \mathcal{L}^0$, see Lemma 5.9 above.

Actually E^0 is the fixed point subspace for the action of $G_p(E)$ on E by shifts. The next lemma says that in characteristic 2 case, $E^0 \neq 0$ whenever $E \neq 0$.

Lemma 5.13. *Suppose that $p = 2$, and let $E \subseteq \mathcal{F}(\Lambda, \bar{K})$ be a non-trivial translation invariant subspace of finite dimension. Then there exists a nonzero function $h \in E^0$ such that the sublattice of periods $\Lambda(h) \subseteq \Lambda$ is characteristic i.e., of odd index.*

Proof. If $2v \in \Lambda$ is a period of a nonzero function $f \in E$ then so is v , maybe, for another such function $h \in E$. Indeed since $f = f_{2v} \neq 0$ then either $f = f_v = h$ or $f \neq f_v$, and then $h = f + f_v \neq 0 \in E$ is v -periodic, as required. In this way we arrive finally at a sublattice $\Lambda'' = \Lambda(h'') \subseteq \Lambda$ of odd index which contains $\Lambda(E)$. This corresponds to passing from the group $G(E) = \Lambda/\Lambda(E)$ to its quotient group $G(E)/G_2(E)$ of odd order, where $G_2(E)$ is the Sylow 2-subgroup of $G(E)$. By Proposition 5.11, $h'' \in E^0$ as required. \square

For any sublattice $\Lambda' \in \mathcal{L}$ the translation invariant subspace

$$(24) \quad E_{\Lambda'} = \mathcal{F}_{\Lambda'}(\Lambda, \bar{K}) = \{f \in \mathcal{F}(\Lambda, \bar{K}) : \Lambda(f) \supseteq \Lambda'\} \subseteq \mathcal{F}(\Lambda, \bar{K})$$

is the maximal such subspace with period lattice Λ' . It is easily seen that $\Lambda'' = \Lambda(E_{\Lambda'}^0)$. The regular representation $\rho : \Lambda \rightarrow \text{Aut}(E_{\Lambda'})$, $v \mapsto \tau_v$, factorizes through a representation $\Lambda \rightarrow S_n$ into the symmetric group, where $n = \text{ind}_{\Lambda}(\Lambda') = \dim_{\bar{K}}(E_{\Lambda'})$ and S_n acts by permutations of the orthonormal basis $((\delta_{v+\Lambda'})_{v \in \Lambda})$ of $E_{\Lambda'}$. The latter one can be identified with the basis of δ -functions in $\mathcal{F}(G, \bar{K}) \cong E_{\Lambda'}$, where $G = \Lambda/\Lambda'$. The representation ρ is induced via $L \rightarrow G$ by the regular representation of G on $\mathcal{F}(G, \bar{K})$. The matrix elements of ρ are the δ -functions δ_g ($g \in G$). Every function $f \in \mathcal{F}(G, \bar{K})$ is a state i.e., a linear combination of matrix elements.

5.4. Examples. We let below $K = \text{GF}(2)$, $\Lambda = \mathbb{Z}^s$, $t = 1$, $a = a^+$, and we denote a_* again by a^+ . A sublattice $\Lambda' \subseteq \Lambda$ will be called \bar{a} -harmonic if there exists a nonzero \bar{a} -harmonic Λ' -periodic function $f : \Lambda \rightarrow \bar{K}$.

Example 5.14. Supposing $s = 2$, for a primitive vector $u_0 = (k, l) \in \Lambda = \mathbb{Z}^2$ we let $\Lambda_0 = \mathbb{Z}v_0$, where $v_0 = (-l, k) \perp u_0$. Since $\gcd(k, l) = 1$ we have $\Lambda/\Lambda_0 \cong \mathbb{Z}$. We let $\pi_0 : \Lambda \twoheadrightarrow \Lambda/\Lambda_0 \cong \mathbb{Z}$, $u \mapsto \langle u, u_0 \rangle$. The induced operator $\Delta_{a_*^+} \in \text{End}(\mathcal{F}(\Lambda/\Lambda_0, K))$ corresponds to the following function on \mathbb{Z} :

$$a_*^+ = \delta_0 + \delta_k + \delta_{-k} + \delta_l + \delta_{-l}.$$

For a function $f \in \mathcal{F}(\Lambda/\Lambda_0, \bar{K})$, one has $\tilde{f} := f(-lx + ky) \in \ker(\Delta_{a^+})$ if and only if f satisfies the equation

$$(25) \quad f(z) + f(z - k) + f(z + k) + f(z - l) + f(z + l) = 0, \quad \forall z \in \mathbb{Z}.$$

In particular if $u_0 = (0, 1)$ then $\tilde{f} \in \ker(\Delta_{a^+}|_{\mathbb{Z}^2}) \iff f \in \ker(\Delta_{a^+} | \mathbb{Z})$. The only a^+ -harmonic sublattice $\Lambda' \in \mathcal{L}^0$ that contains the vector e_1 is $L' = \mathbb{Z}e_1 + 3\mathbb{Z}e_2$ (cf. 5.5 below).

Further, if $u_0 = \pm e_1 \pm e_2$ then $\tilde{f} \in \ker(\Delta_{a^+}) \iff f = 0$. Consequently none of the a^+ -harmonic sublattices $\Lambda' \in \mathcal{L}$ contains a vector of the form $\pm e_1 \pm e_2$.

Next we let $\Lambda' = \mathbb{Z}u_0 + \mathbb{Z}v_0 \subseteq \mathbb{Z}^2 = \Lambda$, where $u_0 = (k, l)$ and $v_0 = (k', l')$ are primitive lattice vectors different from $\pm e_1, \pm e_2, \pm e_1 \pm e_2$. Suppose that $m = \text{ind}_{\Lambda}(\Lambda') = |\det(u_0, v_0)|$ is odd. Then Λ' is a^+ -harmonic if and only if there exists a nonzero solution

$f \in \mathcal{F}(\mathbb{Z}, \bar{K})$ of (25), if and only if there is a primitive m th root of unity $\zeta \in \bar{K}^\times$ satisfying

$$1 + \zeta^k + \zeta^{-k} + \zeta^l + \zeta^{-l} = 0.$$

Such a root ζ determines an a^+ -harmonic character $\theta \in \text{Char}_{a^+ - \text{harm}}(\Lambda/\Lambda_0, \bar{K}^\times)$ of order m , where $\theta : x \mapsto \zeta^x$. It also defines an a^+ -harmonic character $\theta \circ \pi_0 \in \text{Char}_{a^+ - \text{harm}}(\Lambda, \bar{K}^\times)$ and the corresponding sublattice $\Lambda' = \ker(\theta \circ \pi_0)$.

For a finite abelian group $G = \bigoplus_{i=1}^s \mathbb{Z}/n_i\mathbb{Z}$ of odd order we have by 2.2(b)

$$\text{spec}(\Delta_{a^+, G}) = \widehat{a^+}(G^\vee) \subset \bar{K}.$$

Letting $g^\vee \in G^\vee$ be a character with $g^\vee(e_j) = \xi_j = \zeta_j^{k_j}$, where $\zeta_j \in \mu_{n_j}$ is a primitive n_j th root of unity and $0 \leq k_j \leq n_j - 1$, we obtain (cf. (26))

$$\widehat{a^+}(g^\vee) = 1 + \sum_{j=1}^s (g^\vee(e_j) + g^\vee(e_j)^{-1}) = 1 + \sum_{j=1}^s (\zeta_j^{k_j} + \zeta_j^{-k_j}).$$

Therefore

$$\text{CharPoly}(\Delta_{a^+, G}) = \prod_{(k_1, \dots, k_s) \in \mathbb{Z}_{\bar{n}}} \left(x - \left(1 + \sum_{j=1}^s (\zeta_j^{k_j} + \zeta_j^{-k_j}) \right) \right).$$

Examples 5.15. (see [Za₁]) With $\mathcal{V} = (e_1, \dots, e_s)$ being the standard basis in Λ , the following hold.

- A product sublattice $\Lambda_{\bar{n}, \mathcal{V}}$ is a^+ -harmonic if and only if the image \tilde{a}^+ of the symbol

$$(26) \quad \sigma_{a^+, \mathcal{V}} = 1 + \sum_{i=1}^s (x_i + x_i^{-1})$$

is a zero divisor in $K[G]$.

- If $s = 1$ then a sublattice $n\mathbb{Z} \subseteq \mathbb{Z}$ is a^+ -harmonic (equivalently, $1 + x + x^{-1}$ is a zero divisor in $K[x]/(1 + x^n)$) if and only if $n \equiv 0 \pmod{3}$.

- Similarly for every $\bar{n} = (n_1, \dots, n_s) \in \mathbb{N}^s$ with $n_1 \equiv 0 \pmod{3}$, the group $\mathbb{Z}_{\bar{n}}$ is a^+ -harmonic.

- The group $\mathbb{Z}_{5,5} = (\mathbb{Z}/5\mathbb{Z})^2$ is a^+ -harmonic, while $\mathbb{Z}_{7,7} = (\mathbb{Z}/7\mathbb{Z})^2$ is not.

Many more examples of this kind were computed by L. Makar-Limanov⁵ along the same lines, and in [Za₁, Appendix 1] by different methods. Cf. also 5.17 below.

Examples 5.16. ($s = 1$) 1. For $\Lambda' = 2\mathbb{Z} \subseteq \mathbb{Z}$ the representation

$$\rho : \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow S_2 \hookrightarrow \mathbf{GL}(2, K), \quad 1 \mapsto \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{with} \quad \beta^2 = 1$$

is equivalent to

$$\rho' : 1 \mapsto \alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad \alpha^2 = 1.$$

The matrix elements of ρ provide the 2-periodic function $\delta_\Lambda = (\dots 0, 1, 0, 1, 0, 1 \dots)$ on \mathbb{Z} and its shifts, whereas the other states are constant functions.

⁵A letter to the author, 2004, 5p.

2. Similarly for $\Lambda' = 3\mathbb{Z} \subseteq \mathbb{Z}$, ρ factorizes through a faithful representation $\mathbb{Z}/3\mathbb{Z} \rightarrow S_3$. The matrix elements give rise to the shifts of the periodic function

$$\delta_{\Lambda'} = (\dots 0, 0, 1, 0, 0, 1, 0, 0, 1 \dots)$$

on \mathbb{Z} . The a^+ -harmonic function $(\dots 0, 1, 1, 0, 1, 1, 0, 1, 1 \dots)$ and its shifts are states.

3. A cyclic group $G = \mathbb{Z}/m\mathbb{Z}$ is a^+ -harmonic if and only if $m \equiv 0 \pmod{3}$, see 5.15.2. Letting $m = 3l$, $l \in \mathbb{N}$, we fix a primitive m th root of unity $\zeta \in \mu_{3l}$, a primitive cubic root of unity $\omega \in \mu_3$, and we let $g^\vee : ne_1 \mapsto \zeta^n$ be the corresponding character of $G = \mathbb{Z}/3l\mathbb{Z}$. For $\theta = (g^\vee)^t$ one has

$$\theta \in G_{a^+ - \text{harm}}^\vee \iff \zeta^t + \zeta^{-t} = 1 \iff \zeta^t = \omega^{\pm 1} \iff t \equiv \pm l \pmod{3l}.$$

So $\theta = (g^\vee)^l : \mathbb{Z}/3l\mathbb{Z} \rightarrow \mathbb{F}_4^\times$ is an a^+ -harmonic character with trace

$$h = \text{Tr}_{\mathbb{F}_4}(\theta) : ne_1 \mapsto \omega^n + \omega^{2n} = \begin{cases} 0 & \text{if } n \equiv 0 \pmod{3}, \\ 1 & \text{otherwise.} \end{cases}$$

Furthermore

$$d_{a^+, G} = 2, \quad V(a^+) = G_{a^+ - \text{harm}}^\vee = \{\theta, \theta^{-1}\} \quad \text{and} \quad \ker(a^+) = \text{span}(h, h^+),$$

where $h^+(x) = h(1+x)$.

Similarly for $\theta(x) = \omega^x \in \text{Char}_{a^+ - \text{harm}}(\mathbb{Z}, \bar{K}^\times)$, $L(\theta) = 3\mathbb{Z}$ is a maximal proper a^+ -harmonic sublattice of \mathbb{Z} . Moreover h, h^+ lifted to \mathbb{Z} give a basis of $\ker(\Delta_{a^+}|_{3\mathbb{Z}})$.

Example 5.17. ($s = 2$) As another example, we consider the group $G = \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$. We fix a primitive 5th root of unity $\zeta \in \mu_5$. We have $d_{a^+, G} = 8$, see e.g., [Za₁, Appendix 1]. The relation $\zeta + \zeta^2 + \zeta^3 + \zeta^4 = 1$ yields the 8 solutions of (19) with $s = 2$, $t = 1$, $a_1 = a^+$, $\sigma_{a^+} = x_1 + x_1^{-1} + x_2 + x_2^{-1} + 1$ and $n_1 = n_2 = 5$. These solutions can be obtained from $(x_1, x_2) = (\zeta, \zeta^3)$ by suitable transformations. The locus of a^+ -harmonic characters

$$V(a^+) = G_{a^+ - \text{harm}}^\vee = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

consists of two orbits of the cyclic group $\langle D_2 \rangle \cong \mathbb{Z}/4\mathbb{Z}$ acting on G^\vee via $D_2 : g^\vee \mapsto (g^\vee)^2$.

The solution (ζ, ζ^3) ((ζ^3, ζ) , respectively) of (19) gives rise to the a^+ -harmonic character $g^\vee : me_1 + ne_2 \mapsto \zeta^{m+3n}$ (${}^t g^\vee : me_1 + ne_2 \mapsto \zeta^{3m+n}$, respectively), where

$$g^\vee = \begin{pmatrix} 1 & \zeta^3 & \zeta & \zeta^4 & \zeta^2 \\ \zeta & \zeta^4 & \zeta^2 & 1 & \zeta^3 \\ \zeta^2 & 1 & \zeta^3 & \zeta & \zeta^4 \\ \zeta^3 & \zeta & \zeta^4 & \zeta^2 & 1 \\ \zeta^4 & \zeta^2 & 1 & \zeta^3 & \zeta \end{pmatrix} \quad \text{resp.} \quad {}^t g^\vee = \begin{pmatrix} 1 & \zeta & \zeta^2 & \zeta^3 & \zeta^4 \\ \zeta^3 & \zeta^4 & 1 & \zeta & \zeta^2 \\ \zeta & \zeta^2 & \zeta^3 & \zeta^4 & 1 \\ \zeta^4 & 1 & \zeta & \zeta^2 & \zeta^3 \\ \zeta^2 & \zeta^3 & \zeta^4 & 1 & \zeta \end{pmatrix}$$

has trace

$$h = \text{Tr}_{\mathbb{F}_{16}}(g^\vee) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix} \text{ resp. } {}^t h = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

The doubling of periods as in [Za₁, 2.35] produces the following a^+ -harmonic function on $G = (\mathbb{Z}/10\mathbb{Z})^2$:

$$\delta(h) = \begin{pmatrix} 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\ \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This matrix contains five crosses

$$\begin{pmatrix} & \mathbf{1} & & & \\ & \mathbf{1} & & & \\ \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} \\ & \mathbf{1} & & & \\ & \mathbf{1} & & & \end{pmatrix}.$$

Letting $\pi_1 : \Lambda = \mathbb{Z}^2 \twoheadrightarrow (\mathbb{Z}/5\mathbb{Z})^2$ and $\pi_2 : \mathbb{Z}^2 \twoheadrightarrow (\mathbb{Z}/10\mathbb{Z})^2$ it is easily seen that $\Lambda_1 = \Lambda(h \circ \pi_1) = \mathbb{Z}u + \mathbb{Z}v$ and $\Lambda_2 = \Lambda(\delta(h) \circ \pi_2) = 2\mathbb{Z} + 2\mathbb{Z}v$, where $u = (1, 2)$ and $v = (-2, 1)$. Hence $\text{ind}_\Lambda(\Lambda_1) = 5$ and $\text{ind}_\Lambda(\Lambda_2) = 20$. Actually $\Lambda_1 = \Lambda(\theta)$, where $\theta = g \circ \pi_1 \in \text{Char}_{a^+ - \text{harm}}(\Lambda)$. In contrast the sublattice $\Lambda_2 \subseteq \mathbb{Z}^2$ of even index is not characteristic (see Definition 5.8.a), although it is a period lattice for a nonzero harmonic function on $\Lambda = \mathbb{Z}^2$. By virtue of Proposition 5.11, $\delta(h) \circ \pi_2$ is not a linear combination of characters with values in \bar{K}^\times .

6. MULTI-ORDERS TABLE AND PARTNERSHIP GRAPH

In this section we let again $K = \text{GF}(q)$, where $q = p^r$ with $p, r > 0$, so that \bar{K}^\times is a torsion group.

Given a lattice Λ , a base \mathcal{V} of Λ and a system $\Delta_{\bar{a}}$ of convolution operators on Λ , where $\bar{a} = (a_1, \dots, a_t) \in (\mathcal{F}^0(\Lambda, \bar{K}))^t$, we compose a table $\mathcal{D}_{\bar{a}, \mathcal{V}} = \{d_{\bar{a}, \bar{n}, \mathcal{V}}\}_{\bar{n} \in \mathbb{N}^s}$, where

$$d_{\bar{a}, \bar{n}, \mathcal{V}} = \dim \ker (\Delta_{\bar{a}} | \Lambda_{\bar{n}, \mathcal{V}}) = \dim \left(\bigcap_{j=1}^t \ker (\Delta_{a_j} | \Lambda_{\bar{n}, \mathcal{V}}) \right),$$

cf. (5). Thus the entries $d_{\bar{a}, \bar{n}, \mathcal{V}}$ are nonzero for all $\bar{n} \in \mathbb{N}^s$ such that $\Sigma_{\bar{a}, \bar{n}, \mathcal{V}} \neq \emptyset$ (see 3.2). By virtue of 3.1, $\forall \bar{m} = (m_1, \dots, m_s) \in \mathbb{N}_{\text{co}(p)}^s$, $\forall n = (n_1, \dots, n_s) \in \mathbb{N}_{\text{co}(p)}^s$

$$(27) \quad d_{\bar{a}, \text{gcd}(\bar{m}, \bar{n}), \mathcal{V}} \leq \min \{d_{\bar{a}, \bar{m}, \mathcal{V}}, d_{\bar{a}, \bar{n}, \mathcal{V}}\},$$

where $\gcd(\bar{m}, \bar{n}) = (\gcd(m_1, n_1), \dots, \gcd(m_s, n_s))$. Indeed due to (19),

$$\Sigma_{\bar{a}, \gcd(\bar{m}, \bar{n}), \nu} = \Sigma_{\bar{a}, \bar{m}, \nu} \cap \Sigma_{\bar{a}, \bar{n}, \nu}.$$

Remark 6.1. We note that for $s = 1$, the classical Chebyshev-Dickson polynomials T_n of the first kind satisfy the identity $\gcd(T_m, T_n) = T_{\gcd(m, n)}$. Whereas for $s \geq 2$ the inequality (27) is strict in general. For instance for $K = \text{GF}(2)$, $s = 2$, $t = 1$ and $a_1 = a^+$,

$$d_{a^+, (3,3)} = 4 < \min \{d_{a^+, (9,21)}, d_{a^+, (21,9)}\} = 16.$$

Thus the above identity does not hold in general for a Chebyshev-Dickson system.

We consider also the table of multi-orders $\mathcal{S}_{\bar{a}, \nu} = \{s_{\bar{a}, \bar{n}, \nu}\}_{\bar{n} \in \mathbb{N}_{\text{co}(p)}^s}$, where $s_{\bar{a}, \bar{n}, \nu} = \text{card } \Sigma_{\bar{a}, \bar{n}, \nu}$ denotes the number of points $\xi = (\xi_1, \dots, \xi_s)$ on the symbolic hypersurface $\Sigma_{\bar{a}, \nu}$ with multi-order

$$(28) \quad \bar{n} = \text{multi-ord}(\xi) = (\text{ord}(\xi_1), \dots, \text{ord}(\xi_s)) \in \mathbb{N}_{\text{co}(p)}^s.$$

These two tables $\mathcal{D}_{\bar{a}, \nu}$ and $\mathcal{S}_{\bar{a}, \nu}$ are related via

$$d_{\bar{a}, \bar{n}, \nu} = \sum_{\bar{d} | \bar{n}} s_{\bar{a}, \bar{d}, \nu},$$

where $\bar{n} \in \mathbb{N}_{\text{co}(p)}$ and \bar{d} runs over all s -tuples $\bar{d} = (d_1, \dots, d_s) \in \mathbb{N}_{\text{co}(p)}^s$ with $d_i \mid n_i \forall i = 1, \dots, s$.

Letting $\bar{n} = (n_1, \dots, n_s) = (\bar{n}', n_s)$, for any fixed $\bar{\xi}' = (\xi_1, \dots, \xi_{s-1}) \in \mu_{\bar{n}'} = \bigoplus_{i=1}^{s-1} \mu_{n_i}$ the system $\sigma_{a_j}(\bar{\xi}', x_s) = 0$, $j = 1, \dots, t$, has a finite set of solutions $x_s = \eta \in \bar{K}^\times$. Hence for any \bar{n}' the line $(s_{\bar{a}, (\bar{n}', n_s), \nu})$ of the table $\mathcal{S}_{\bar{a}, \nu}$ has bounded support in \mathbb{Z} .

We let

$$l(\bar{n}') = \text{lcm}(\text{ord}(\eta) : \exists \bar{\xi}' \in \mu_{\bar{n}'}, \sigma_{a_j}(\bar{\xi}', \eta) = 0 \quad \forall j = 1, \dots, t).$$

Hence for any \bar{n}' the line $(d_{\bar{a}, (\bar{n}', n_s), \nu})_{n_s}$ of the table $\mathcal{D}_{\bar{a}, \nu}$ is periodic with minimal period $l(\bar{n}')$:

$$d_{\bar{a}, (\bar{n}', n_s + l(\bar{n}')), \nu} = d_{\bar{a}, (\bar{n}', n_s), \nu} \quad \forall \bar{n}' \in \mathbb{N}_{\text{co}(p)}^{s-1}.$$

The set $(l(\bar{n}'))_{\bar{n}' \in \mathbb{N}_{\text{co}(p)}^{s-1}}$ of all such periods is in general unbounded.

For instance for $K = \text{GF}(2)$, $s = 2$, $t = 1$ and $a_1 = a^+$ we have

$$\max_{m \in \mathbb{N}_{\text{odd}}} \{d_{a^+, (n, m)}\} = d_{a^+, (n, l(n))} = \begin{cases} 2n, & n \not\equiv 0 \pmod{3} \\ 2n - 2, & \text{otherwise.} \end{cases}$$

Following [Za₁] we let

$$\mathcal{E}_{\bar{a}, \nu} = \{\bar{n} \in \mathbb{N}_{\text{co}(p)}^s \mid d_{\bar{a}, \bar{n}, \nu} \neq 0\} \quad \text{and} \quad \mathcal{E}_{\bar{a}, \nu}^0 = \{\bar{n} \in \mathbb{N}_{\text{co}(p)}^s \mid s_{\bar{a}, \bar{n}, \nu} \neq 0\}.$$

By 3.2, $\mathcal{E}_{\bar{a}, \nu}^0 \subseteq \mathcal{E}_{\bar{a}, \nu}$. Letting $\bar{k}\bar{n} = (k_1 n_1, \dots, k_s n_s)$ we obtain a natural covering $\pi : \Lambda / \Lambda_{\bar{k}\bar{n}, \nu} \rightarrow \Lambda / \Lambda_{\bar{n}, \nu}$. Any \bar{a}_* -harmonic function on the second group lifts to such a function on the first one. Therefore $\mathcal{E}_{\bar{a}, \nu}$ is generated by $\mathcal{E}_{\bar{a}, \nu}^0$ as an $\mathbb{N}_{\text{co}(p)}^s$ -module. So in order to determine $\mathcal{E}_{\bar{a}, \nu}$ it is enough to determine $\mathcal{E}_{\bar{a}, \nu}^0$.

Assuming that the symbols σ_{a_j} , $j = 1, \dots, t$, are symmetric i.e., stable under the natural action of the symmetric group S_s on the Laurent polynomial ring $\bar{K}[x_1, x_1^{-1}, \dots, x_s, x_s^{-1}]$, it is possible to replace the multi-orders table $\mathcal{S}_{\bar{a}, \nu}$ by the labelled 'partnership hypergraph' $\mathcal{P}_{\bar{a}}$. The latter one has the set of naturals $\mathbb{N}_{\text{co}(p)}$ as the set of vertices, and

consists of all $(s - 1)$ -simplices $\bar{n} \in \mathcal{E}_{\bar{a}, \nu}^0$ labelled with $s_{\bar{a}, \bar{n}, \nu} \neq 0$. For $s = 2$, $\mathcal{P}_{\bar{a}}$ is just the infinite labelled graph with the set of vertices $\mathbb{N}_{\text{co}(p)}$ and with the edges $[m, n] \in \mathcal{E}_{a^+, \nu}^0$ labelled with $s_{a^+, (m, n), \nu} \neq 0$.

Example 6.2. As was observed by Don Zagier, in the case where $s = 2$, $K = \text{GF}(2)$, $t = 1$ and $a_1 = a^+$, all connected components of the partnership graph \mathcal{P}_a are finite; see [Za₁, Theorem 3.11]. Actually every such component is contained in a level set $f_0^{-1}(r)$ of the suborder function

$$f_0(n) = \text{sord}_n(2) = \min\{j : 2^j \equiv \pm 1 \pmod{n}\}$$

(see e.g. [MOW]). Indeed for a primitive n th root of unity $\zeta \in \mu_n$, by 5.10(a) in [Za₁] one has $f_0(n) = \deg(\zeta + \zeta^{-1})$. Let $(\zeta, \eta) \in \sigma_{a^+}$ be a point with bi-order $(\text{ord}(\zeta), \text{ord}(\eta)) = (m, n)$. Then (ζ, η) satisfies the symbolic equation

$$\zeta + \zeta^{-1} + \eta + \eta^{-1} = 1.$$

Hence $K(\zeta + \zeta^{-1}) = K(\eta + \eta^{-1})$ and so $f_0(m) = f_0(n)$.

It is plausible [Za₁, 4.1] that the connected components of the partnership graph \mathcal{P}_{a^+} coincide with the corresponding level sets of the suborder function f_0 except for $r = 5$ ($f_0^{-1}(5)$ consists of two such components). This conjecture is based on the computation of the first 13 of these components done by Don Zagier with PARI, see Appendix 1 in [Za₁].

The following questions arise.

Problems.

- Given a lattice Λ of rank $s = 2$ and a Galois field $K = \text{GF}(q)$, describe the set of all functions $a \in \mathcal{F}^0(\Lambda, K)$ such that all components of the graph \mathcal{P}_a are finite.
- Is it possible to reconstruct the function a starting from the graph \mathcal{P}_a ?
- Determine the set of all functions $a \in \mathcal{F}^0(\Lambda, K)$ such that the irreducible factors of the Chebyshev-Dickson polynomials $\text{CharPoly}_{a, \bar{n}, \nu}$ exhaust all irreducible polynomials over K . This is indeed the case for the classical Chebyshev-Dickson polynomials T_n , see e.g. [HMP, 2.8-2.10] and the references therein.
- ⁶ Does there exist any reasonable multivariate generating function for the tables $\mathcal{D}_{\bar{a}, \nu}$ or $\mathcal{S}_{\bar{a}, \nu}$? The natural analogs of the logarithm of the Weil zeta function is unlikely to play this role.

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⁶This question is proposed by Roland Bacher in the case of σ^+ -automata.

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